Counting the compositions of a positive integer n using Generating Functions

Start with,

$$\frac{1}{1-\mathrm{x}} = x + x^2 + x^3 + x^4 + \ldots.$$

Where , for example, the co-eff of x^4 is 1, for one summand composition of 4 namely, 4.
To obtain number of compositions of n, we need the co-eff of x^n in

$$\left(x + x^2 + x^3 + \ldots.\right)^2 = \left[\frac{x}{(1-x)}\right]^2 = \frac{x^2}{(1-x)^2}$$

Here for instance we obtain x^4 in $(x+x^2+x^3+x^4+\ldots)^2$ from products $(x^1.x^3)$, $(x^2.x^2)$, and $(x^3.x^1)$. So co-eff of x^4 in $x^2/(1-x)^2$ is 3, which is number of two summand compositions of 4), 1+3, 2+2, 3+1.

Continuing with the three summand compositions we now examine

$$\left(x + x^2 + x^3 + x^4 + \ldots\right)^3 = \left[\frac{x}{(1-x)}\right]^3 = \frac{x^3}{(1-x)^3}$$

Once again we look at the ways x^4 comes about – namely, from products $(x^1.x^1.x^2)$, $(x^1.x^2.x^1)$, and $(x^2.x^1.x^1)$. So here co-eff of x^4 is 3, which accounts for the three summand compositions 1+1+2, 1+2+1, and 2+1+1 (of 4).
Finally the co-eff of x^4 in below function is 1,

$$\left(x + x^2 + x^3 + x^4 + \ldots\right)^4 = \left[\frac{x}{(1-x)}\right]^4 = \frac{x^4}{(1-x)^4}$$ for one four summand composition

1+1+1+1 (of 4).
These result tell us that the co-eff of x^4 in

$$\sum_{i=1}^{4}\left[\frac{x}{(1-x)}\right]^i$$ is 1+3+3+1 = 8 (=2^3), the number of compositions of 4. In fact this is

also the co-eff of x^4 in the above equn.
Generalizing the situation we find that the number of compositions of a positive integer n is the co-eff of x^n in the generating function

$$f(x) = \sum_{i=1}^{\infty}\left[\frac{x}{(1-x)}\right]^i \quad \ldots\ldots\ldots(1).$$

But if we set y=x/(1-x), then it follows that

$$f(x)=\sum_{i=1}^{\infty} y^i = y\sum_{i=0}^{\infty} y^i = y\left(\frac{1}{1-y}\right) = \left(\frac{\text{x}}{(1-\text{x})}\right)\left(\frac{1}{1-\left(\frac{x}{1-x}\right)}\right)$$

$$=\left(\frac{\text{x}}{(1-\text{x})}\right)\left(\frac{1}{\frac{1-x-x}{1-x}}\right)$$

$$=\frac{x}{(1-2x)} = \text{x}\left[1+(2\text{x})+(2x)^2+(2x)^3+\ldots\right]$$

$$=2^0x+2^1x^2+2^2x^3+2^3x^4+\ldots.$$

So the number of integer compositions of a positive integer n is the co-eff of x^n in f(x) and this is 2^{n-1} as derived in the equation in previous slide.

Let us examine the identity

$\left(\frac{1-x^{n+1}}{1-x}\right)=1+x+x^2+x^3+\ldots..+x^n$ When x is replaced by 2 in this the result tells that for all n belonging to Z^+,

$1+2+2^2+2^3+\ldots..+2^n=\left(\frac{1-2^{n+1}}{1-2}\right)=2^{n+1}-1$. Where do we use this?

Consider the special compositions of integers 6 and 7, that read same left to right as right to left.

	6				7	
	1+4+1				**1+5+1**	
	2+2+2				**2+3+2**	
	1+1+2+1+1				**1+1+3+1+1**	
	3+3				**3+1+3**	
	1+2+2+1				**1+2+1+2+1**	
	2+1+1+2				**2+1+1+1+2**	
	1+1+1+1+1+1				**1+1+1+1+1+1+1**	

These are palindromes for 6 and 7. We find that for 7 there are 1+(1+2+4) = 1+(1+2^1+2^2) = 1+(2^3-1) = 2^3 palindromes. There is one palindrome with one summand, 7. There is also one palindrome where center summand is 5 and where we place one composition of 1 on either side of this summand (palindrome 2).

For the center summand 3 we place one of the two compositions of 2 on the right and then match it on the left, with same composition, in reverse order. (palindromes 3 and 4) finally when the center summand is 1, we put a given composition of 3 on the right side f this 1 and match on left side with same composition, in reverse order. There are 2^{3-1} = 4 compositions of 3 (palindromes 5,6,7,8).

The situation is same for palindromes of 6 except case where + sign appears as center. So for n=6, i)Center summand 6 — 1 palindrome

ii)Center summand 4 — $1(=2^{1-1})$ palindrome

iii) Center summand 2 — $2(=2^{2-1})$ palindrome

iv) + sign at Center — $4(=2^{3-1})$ palindrome

So there are $1+(1+2^1+2^2) = 1+(2^3-1)=2^3$ palindromes for 6.

Now we look at the general situation. For n=1 there is one palindrome. If n = 2k+1, for k belonging to Z^+, then there is one palindrome with center summand n. for $1 \leq t \leq k$, there are 2^{t-1} palindromes of n with center summand n-2t. Hence the total number of palindromes of n is

$$1+(1+2^2+2^3+\ldots\ldots+2^{k-1}) = 1+(2^k-1) = 2^k = 2^{(n-1)/2}$$

Now consider n even, say n = 2k for k belonging to Z^+.

Here there is one palindrome with center summand n-2s (one palindrome for each of 2^{s-1} compositions of s). In addition there are 2^{k-1} palindromes where a + sign is at the center (one palindrome for each of the 2^{k-1} compositions of k).In total, n has

$$1+(1+2^1+2^2+2^3+\ldots\ldots+ 2^{k-2}+2^{k-1}) = 1+(2^k-1) = 2^k = 2^{n/2}$$

Observe that for $n \in Z^+$, n has $2^{\lfloor n/2 \rfloor}$ palindromes.

Partitions of Integers Partition a positive integer *n* into positive summands and seeking the number of such partitions, without regard to order. This number is denoted by p(n).

For example, $p(1)=1$: 1

$p(2)=2$: 2=1+1

$p(3)=3$: 3=2+1=1+1+1

$p(4)=5$: 4=3+1=2+2=2+1+1=1+1+1+1

$p(5)=7$: 5=4+1=3+2=3+1+1=2+2+1=2+1+1+1
=1+1+1+1+1

We should like to obtain $p(n)$ for a given n without having to list all the partitions. We need a tool to keep track of the numbers of 1's, 2's, ..., n's that are used as summands for n.

$$\text{keep track of 1's}: 1+x+x^2+x^3+\cdots$$

$$\text{keep track of 2's}: 1+x^2+x^4+x^6+\cdots$$

$$\vdots$$

$$\text{keep track of } k\text{'s}: 1+x^k+x^{2k}+x^{3k}+\cdots$$

For example, p(10) is the coefficient of x^{10} in

$$f(x) = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots(1+x^{10}+\cdots)$$

$$=\frac{1}{(1-x)}\frac{1}{(1-x^2)}\frac{1}{(1-x^3)}\cdots\frac{1}{(1-x^{10})}=\prod_{i=1}^{10}\frac{1}{(1-x^i)}.$$

In general, $P(x)=\prod_{i=1}^{\infty}\frac{1}{1-x^i}$ generate the sequence $p(0), p(1), \cdots$

Example: Find the generating function for the number of ways an advertising agent can purchase n minutes of air time if time slots for commercials come in blocks of 30, 60, or 120 seconds.

Let 30 seconds represent one time unit. Then the answer is the number of integer solutions to the equation $a + 2b + 4c = 2n$ with $0 \le a, b, c$. The associated generating function is

$$f(x) = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^4 + x^8 + \cdots) =$$

$\frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^4}$ and the coefficient of x^{2n} is the answer.

Example: Find the generating function for $p_d(n)$, the number of partitions of a positive integer n into distinct summands.

Let us consider 11 partitions of 6:

1)1+1+1+1+1+1 2)1+1+1+1+2 3)1+1+1+3
4)1+1+4 5)1+1+2+2 6)1+5
7)1+2+3 8)2+2+2 9)2+4
10)3+3 11)6

Partitions 6,7,9 and 11 have distinct summands, so $P_d(6)=4$

For any $k \in Z^{+}$, There are two posiibilities

either k is not used as a summand or it is.

This can be accounted for by the polynomial $1+x^{k}$.

Consequently, the generating function is

$$P_d(x) = (1+x)(1+x^2)(1+x^3)\cdots = \prod_{i=1}^{\infty}(1+x^i)$$

for each $n \in Z^{+}$, $p_d(n)$ is the coeff of x^n in

$(1+x)(1+x^2)\ldots\ldots.(1+x^n)$. and $p_d(0)=1$.

when $n=6$, the coeff of x^6 *in* $(1+x)(1+x^2)\ldots\ldots.(1+x^6)$ is 4.

Considering the partitions, we see that there are four partitions of 6 into odd summands, namely 1, 3, 6 and 10 in the previous example. We also have $p_d(6)=4$.

let $p_0(n)$ denote the number of partitions of n into odd summands, when $n \geq 1$. We define $p_0(0)=1$. The generating function for the sequence $p_0(0), p_0(1), p_0(2), \ldots.$ is given by

$$P_0(x) = (1+x+x^2+x^3+\ldots..)(1+x^3+x^6+\ldots.)(1+x^5+x^{10}+\ldots).$$

$$(1+x^7+x^{14}+\ldots) = \frac{1}{1-x}.\frac{1}{1-x^3}.\frac{1}{1-x^5}.\frac{1}{1-x^7}\ldots..$$

Now because,

$$1+x = \frac{1-x^2}{1-x}, \quad 1+x^2 = \frac{1-x^4}{1-x^2}, \quad 1+x^3 = \frac{1-x^6}{1-x^3}, \quad \ldots..$$

we have,

$$P_d(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)\ldots.$$

$$= \frac{1-x^2}{1-x}.\frac{1-x^4}{1-x^2}.\frac{1-x^6}{1-x^3}.\frac{1-x^8}{1-x^4}\ldots.$$

$$= \frac{1}{1-x}.\frac{1}{1-x^3}.\frac{1}{1-x^5}\ldots.$$

$$= P_0(x)$$

From equality of generating functions,

$$p_d(n) = p_0(n), \text{ for all } n \geq 0.$$

Example: Partition into odd summands but each such odd summands must occur an odd umber of times-or not at all. Here, for example, there is one such partition of integer 1, namely, there are no partitions of 2, there are two such partitions for integer 3, namely 3 and 1+1+1. one partition for integer 4 namely 3+1. The generating function for the partitions described is given by

$$f(x) = (1 + x + x^3 + x^5 + \cdots)(1 + x^3 + x^9 + x^{15} + \cdots)(1 + x^5 + x^{15} + \cdots)\cdots = \prod_{k=0}^{\infty}\left(1 + \sum_{i=0}^{\infty} x^{(2k+1)(2i+1)}\right).$$

Using Generating functions, we will also be able to deal with a sample space that is discrete but not finite.

Example: Suppose that Brianna takes an examination until she passes it. Further, suppose the probability that she passes the examinations on any given attempt is 0.8 and the result of each attempt, after the first, is independent of any previous attempt. If we let P denote "pass" and F denote "fail", for any given attempt, then our sample space may be expressed as

= {P, FP, FFP, FFFP,….}Where, for example, Pr(FFP) is the probability that she fails the exams is twice before she passes it, which is given by $(0.2)^2(0.8)$. In addition, the sum of probabilities for the outcomes in is Now suppose we want to know the probability she passes the exam on an even numbered attempt. That is we want Pr(A) where A is the event {FP, FFFP, ….}.

At this point we introduce the discrete random variable Y where Y counts the number of attempts up to and including the one where she passes the exam. Then the probability distribution for Y is given by $\Pr(Y=y) = (0.2)^{y-1}(0.8)$, $y \geq 1$.

So Pr(A) can be determined as follows:

$$\begin{aligned}
\Pr(A) = \sum_{i=1}^{\infty} \Pr(y = 2i) = \sum_{i=1}^{\infty} (0.2)^{2i-1}(0.8) &= (0.8)\sum_{i=1}^{\infty}(0.2)^{2i-1} \\
&= (0.8)\left[(0.2) + (0.2)^3 + (0.2)^5 + \ldots\right] \\
&= (0.8)(0.2)\left[1 + (0.2)^2 + (0.2)^3 + (0.2)^4 + \ldots\right] \\
&= (0.8)(0.2)\frac{1}{1-(0.2)^2} \\
&= \frac{(0.8)(0.2)}{0.96} = \frac{1}{6}
\end{aligned}$$

Continuing with Y, now we'd like to find E(Y), the number of time she expects to take exam before she passes it. To determine E(Y) we'll start with the formula,

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + ...$$

taking the derivative both sides, we find that

$$(-1)(1-t)^{-2}(-2) = \frac{1}{(1-t)^2} = \frac{d}{dt}\left[\frac{1}{1-t}\right]$$

$$= 1 + 2t + 3t^2 + 4t^3 + ...$$

where this series converges for $|t| < 1$.

Therefore,

$$E(Y) = \sum_{y=1}^{\infty} y.\Pr(Y = y) = \sum_{y=1}^{\infty} y(0.2)^{y-1}(0.8)$$

$$= (0.8)\sum_{y=1}^{\infty} y(0.2)^{y-1}$$

$$= (0.8)\left[1 + 2(0.2) + 3(0.2)^2 + +4(0.2)^3 + ...\right]$$

$$= (0.8)\frac{1}{(1-0.2)^2} = \frac{0.8}{(0.8)^2}$$

$$= \frac{1}{0.8} = 1.25$$

so she expects to take exam 1.25 times before she passes it.

Finally, to determine Var(Y), we find first $E(Y^2)$.

To do so miltiply by t the differentiated previous result.

then,

$$\frac{t}{(1-t)^2} = t + 2t^2 + 3t^3 + 4t^4 + ...$$

Differentiate both sides, now we get,

$$\frac{(1-t)^2(1) - t(2)(1-t)(-1)}{(1-t)^4} = \frac{1+t}{(1-t)^3} = \frac{d}{dt}\left[\frac{t}{(1-t)^2}\right]$$

$$= 1^2 + 2^2 t + 3^2 t^2 + 4^2 t^3 + ...$$

and this aslo converges for $|t| < 1$.

So now we have,

$$E(Y^2) = \sum_{y=1}^{\infty} y^2 \Pr(Y = y) = \sum_{y=1}^{\infty} y^2 (0.2)^{y-1} (0.8)$$

$$= (0.8)\sum_{y=1}^{\infty} y^2 (0.2)^{y-1}$$

$$= (0.8)\left[1^2 + 2^2 + (0.2) + 3^2 (0.2)^2 + 4^2 (0.2)^3 + ...\right]$$

$$= (0.8)\left[\frac{1 + 0.2}{(1 - 0.2)^3}\right]$$

$$= \frac{1.2}{(0.8)^2} = \frac{15}{8}$$

Consequently,

$$\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2$$

$$= \frac{15}{8} - \left(\frac{5}{4}\right)^2$$

$$= \frac{30 - 25}{16}$$

$$= \frac{5}{16}$$

Exponential Generating Functions:

The generating functions we have dealt now are called ordinary Generating functions, which arose in selection problems where order was irrelevant. Now let us turn to the problems where order is relevant and crucial. We seek a tool. To find such a tool let us consider the binomial theorem. For each n belongs to Z^+,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots.. + \binom{n}{n}x^n,$$

so $(1+x)^n$ is the ordinary generating function for the sequence, When dealing with

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots.., \binom{n}{n}, 0,0,0\ldots$$

this we wrote that C(n,r) represented the number of combinations of n objects taken r at a time with $0 \le r \le n$. Consequently $(1+x)^n$ generated the sequence C(n,0), C(n,1), C(n,2), C(n,3),….., C(n,n)

Now for all $0 \le r \le n$,

$$C(n,r) = \frac{n!}{r!(n-r)!} = \left(\frac{1}{r!}\right)P(n,r),$$

where P(n, r) denotes the permutations of n objects taken r at a time.So,

$$(1+x)^n = C(n,0) + C(n,1)x + C(n,2)x^2 + C(n,3)x^3 + \ldots + C(n,n)x^n$$

$$= P(n,0) + P(n,1)x + P(n,2)\frac{x^2}{2!} + P(n,3)\frac{x^3}{3!} + \ldots.. + P(n,n)\frac{x^n}{n!}.$$

On the basis of this observation We have the following definition.

For a sequence $a_0, a_1, a_2, a_3, a_4, a_5, \ldots..$ of real numbers,

$$f(x) = a_0 + a_1 x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \ldots. = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!},$$

is called the exponential generating function for the given sequence.

Eg : The Maclaurian series expansion for e^x is,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots.. = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$$

so e^x is the exponential generating function for the sequence 1,1,1,1,1,.......

The function e^x is the ordinary generating function for the sequence,

$$1, 1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \frac{1}{5!}, \ldots..$$

Example: In how many ways can four letters of ENGINE be arranged?

The following table shows list of possible selections of size 4 from the letters E,N,G,I,N,E, along with number of arrangements those 4 letters determine.

E E N N	4!/(2!2!)	E G N N	4!/2!
E E G N	4!/2!	E I N N	4!/2!
E E I N	4!/2!	G I N N	4!/2!
E E G I	4!/2!	E I G N	4!

Let us obtain the solution by using exponential gen. fun.

For the letter E we use $[1+x+(x^2/2!)]$, because there are 0, 1 or 2 E's to arrange. The number of distinct ways to arrange two E's is 1 (co-eff of the term $x^2/2$).For the letter N we use $[1+x+(x^2/2!)]$, because There are 0, 1 or 2 N's to arrange. The number of distinct ways to arrange two N's is 1 (co-eff of the term $x^2/2$).The arrangements for each of the letters G and I are represented by (1+x).Consequently, the exponential generating function is,

$f(x) = \left[1 + x + \left(x^2/2!\right)\right]^2 (1+x)^2$ the answer is co - eff of $x^4/4!$Consider two of the eight ways in which the term $x^4/4!$ arises in the expansion of

$$f(x) = \left[1 + x + \left(x^2/2!\right)\right]\left[1 + x + \left(x^2/2!\right)\right](1+x)(1+x)$$

$$\left(x^2/2!\right)\left(x^2/2!\right)(1)(1)$$

1) From the product where $\left(x^2/2!\right)$ is taken from first two factors and 1 is taken from last two factors.

Then

$$\left(x^2/2!\right)\left(x^2/2!\right)(1)(1)$$

$$= \left(x^4/2!2!\right)(1)(1)$$

$$= (4!/2!2!).\left(x^4/4!\right)$$

And the co-eff of $x^4/4!$ is 4!/(2!2!) which is the number of ways one can arrange four letters E, E, N, N.

2) From the product

$$\left(x^2/2!\right)(1)(x)(x)$$

where $\left(x^2/2!\right)$ is taken from first factor, 1 is taken from second factor and x is taken from last two factors.

Here

$(x^4/2!)(1)(x)(x) = x^4/2! = (4!/2!)(x^4/4!)$

So the co-eff of $x^4/4!$ is 4!/2! Which is the number of ways the four letters E, E, G, I can be arranged.In the complete expansion of the f(x), the term involving x^4

$$\left(\frac{x^4}{2!2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + \frac{x^4}{2!} + x^4\right)$$

and consequently $x^4/4!$, is

$$= \left[\left(\frac{4!}{2!2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + \left(\frac{4!}{2!}\right) + 4!\right]\left(\frac{x^4}{4!}\right)$$

Where the co-eff of $x^4/4!$ Is the answer (102 arrangements) produced by the eight results in the table

Example: Consider the Maclaurian series expansions of e^x and e^{-x}

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \qquad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$$

add these series together we get,

$$e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots.\right)$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots.$$

subtract the series we get

$$\frac{e^x - e^{-x}}{2} = 1 + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots.$$

These results help us in following examples

Example: A ship carries 48 flags, 12 each of the colors red, blue, white and black. 12 of these flags are placed on a vertical pole in order to communicate a signal to other ships.

a) How many of these signals use an even number of blue flags and an odd number of black flags?

Exponential generating function,

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)^2 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right)\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)$$

considers all signal made up of n flags, $n \geq 1$. The last two factors restrict to even no. of blue and odd no. of black flags.

Since,

$$f(x) = \left(e^x\right)^2 \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \left(\frac{1}{4}\right)\left(e^{2x}\right)\left(e^{2x} - e^{-2x}\right) = \frac{1}{4}\left(e^{4x} - 1\right)$$

$$= \frac{1}{4}\left(\sum_{i=0}^{\infty} \frac{(4x)^i}{i!} - 1\right) = \frac{1}{4}\left(\sum_{i=1}^{\infty} \frac{(4x)^i}{i!}\right)$$

The co-eff of $x^{12}/12!$ in f(x) yields $(1/4)(4^{12})=4^{11}$ signals made up of 12 flags with even no. blue & odd no. black flags

b) How many of the signals have at least 3 white flags, or no white flags at all?

Exponential generating function,

$$f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\left(1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)^2$$

$$= e^x\left(e^x - x - \frac{x^2}{2!}\right)(e^x)^2 = e^{3x}\left(e^x - x - \frac{x^2}{2!}\right)$$

$$= e^{4x} - xe^{3x} - \left(\frac{1}{2}\right)x^2 e^{3x}$$

$$= \sum_{i=0}^{\infty}\frac{(4x)^i}{i!} - x\sum_{i=0}^{\infty}\frac{(3x)^i}{i!} - \left(\frac{x^2}{2}\right)\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}$$

Here the factor,

$$\left(1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) = e^x - x - \frac{x^2}{2!}$$

restricts the signals to those that contain three or more of the 12 white flags, or none at all.

The answer for the no. signals here is the co-eff of $x^{12}/12!$ in f(x). As we consider each summand, we find

i) $\sum_{i=0}^{\infty}\frac{(4x)^i}{i!}$, here we have a term $\frac{(4x)^{12}}{12!} = 4^{12}\left(\frac{x^{12}}{12!}\right)$,

so the co-eff of $\frac{x^{12}}{12!}$ is 4^{12}.

ii) $x\left(\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}\right)$. In this, in order to consider the term $x^{12}/12!$,

we need to consider the term

$$x\left[\frac{(3x)^{11}}{11!}\right] = 3^{11}\left[\frac{(x)^{12}}{11!}\right] = (12)(3)^{11}\left[\frac{(x)^{12}}{12!}\right]$$

and here the co-eff of $\left[\frac{(x)^{12}}{12!}\right]$ is $(12)(3)^{11}$.

iii) $\left(\frac{x^2}{2}\right)\left(\sum_{i=0}^{\infty}\frac{(3x)^i}{i!}\right)$, for this last summand, in order to get term $\frac{x^{12}}{12!}$,

we need to consider the term

$$\left(\frac{x^2}{2}\right)\left[\frac{(3x)^{10}}{10!}\right]=\left(\frac{1}{2}\right)(3x)^{10}\left(\frac{x^{12}}{10!}\right)=\left(\frac{1}{2}\right)(12)(11)(3)^{10}\left(\frac{x^{12}}{12!}\right),$$

where the co - eff of $\frac{x^{12}}{12!}$ is $\left(\frac{1}{2}\right)(12)(11)(3)^{10}$.

consequently, the number of 12 flag signals with at least 3 white flags, or none at all, is

Result of i + Result of ii + Result of iii

$$4^{12}+12\left(3^{11}\right)+\left(\frac{1}{2}\right)(12)(11)(3)^{10}=10{,}754{,}218.$$

Example: Company hires 11 new employees, each of whom is to be assigned to one of the four subdivisions. Each subdivision will get at least one new employee. In how many ways can these assignments be made?

Calling the subdivisions A, B, C and D, we can equivalently count the 11 letter sequences in which there is at least one occurrence of each letters A, B, C, and D. The exponential generating function for these arrangement is:

$$f(x)=\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots\right)^4$$

$$f(x)=\left(e^x-1\right)^4$$

$$=e^{4x}-4e^{3x}+6e^{2x}-4e^x+1$$

the answer is the co - eff of $\frac{x^{11}}{11!}$ in f(x):

$$4^{11}-4(3)^{11}+6(2)^{11}-4(1)^{11}$$

$$=\sum_{i=0}^{4}(-1)^i\binom{4}{i}(4-i)^{11}$$

Example: Determine the sequences generated by following exponential generating

a) $f(x) = 5e^{5x}$.

$$\text{soln}: f(x) = 5e^{5x} = 5\sum_{n=0}^{\infty}\frac{(5x)^n}{n!}$$

this produces the sequence $5, 5^2, 5^3, 5^4, ...$

functions. b) $f(x) = 7e^{8x} - 4e^{3x}$

$$\text{soln}: f(x) = 7e^{8x} - 4e^{3x} = 7\sum_{n=0}^{\infty}\frac{(8x)^n}{n!} - 4\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}$$

the sequence is $7(8)^n - 4(3)^n$ with $n = 0,1,2,3....$

i.e 3, 44, 412, 3476,....

c) $f(x) = 2e^{x} + 3x^2$

$$\text{soln}: f(x) = 2e^{x} + 3x^2 = 2\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) + 3x^2$$

so the sequence is $2, 2, (2+3), 2, 2, 2, ...$

d) $f(x) = e^{3x} - 28x^3 + 6x^2 + 9x$

$$\text{soln}: f(x) = e^{3x} - 28x^3 + 6x^2 + 9x$$

$$= \left(\sum_{n=0}^{\infty}\frac{3^n x^n}{n!}\right) - 28x^3 + 6x^2 + 9x$$

so sequence is $3^0, (3^1 + 9), (3^2 + 6), (3^3 - 28), 3^4, ...$

which is 3,12,3,-1,91,....

Summation Operator

In this section we introduce a technique that helps us to go from ordinary generating function for sequence a_0, a_1,a_2, a_3, …. to generating function for the sequence a_0, a_0+a_1, $a_0+a_1+a_2$, $a_0+a_1+a_2+a_3$, ….

for $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$, consider, the function $\frac{f(x)}{(1-x)}$.

$$\frac{f(x)}{(1-x)} = f(x).\frac{1}{(1-x)}$$
$$= \left[a_0 + a_1x + a_2x^2 + a_3x^3 + ...\right]\left[1 + x + x^2 + x^3 + ...\right]$$
$$= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + (a_0 + a_1 + a_2 + a_3)x^3 + ...$$

so, $\frac{f(x)}{(1-x)}$ generates the sequence $a_0, (a_0 + a_1), (a_0 + a_1 + a_2), ...$

Thus we refer to $\frac{1}{(1-x)}$ as summation operator.

We know that $\frac{1}{1-x}$ is the gen. fun. for the sequence 1,1,1,...

Apply the summation operator $\frac{1}{1-x}$, we get,

$\frac{1}{1-x}.\frac{1}{1-x}$ is the gen. fun. for the sequence $1, 1+1, 1+1+1, ...$

$\frac{1}{1-x}.\frac{1}{1-x} = \frac{1}{(1-x)^2}$ is the gen. fun. for the sequence 1,2,3,4... .

Consider the gen.fun. $x + x^2$, for the sequence 0,1,1,0,0,0...

Apply the summation operator, we get,

$(x + x^2)\left(\frac{1}{1-x}\right) = \frac{x + x^2}{1-x}$ which is gen. fun for $0, 0+1, 0+1+1, 0+1+1+1, ...$

i.e the sequence 0,1,2,3,4,...

Apply again the summation operator, we get,

$\left(\frac{x + x^2}{1-x}\right)\left(\frac{1}{1-x}\right) = \frac{x + x^2}{(1-x)^2}$ which is gen.fun. for the sequennce

$0, 0+1, 0+1+2, 0+1+2+2, ...$ ie, 0,1,3,5,....

Apply again the summation function, we get,

$$\frac{x+x^2}{(1-x)^2}\left(\frac{1}{1-x}\right)=\frac{x+x^2}{(1-x)^3}$$ which is the gen. fun. for

the sequence $0, 0+1, 0+1+3, 0+1+3+5, \ldots.$

i.e $0, 1, 4, 9, \ldots..$

This suggests that for $n \geq 1$, $\sum_{k=1}^{n}(2k-1)=n^2$

Example: Find a formula to express $0^2 + 1^2 + 2^2 + 3^2 + \ldots + n^2$ as a function of n.

We start with

$$g(x)=\frac{1}{1-x}=1+x+x^2+x^3+\ldots. \quad \text{then,}$$

$$(-1)(1-x)^{-2}(-1)=\frac{1}{(1-x)^2}=\frac{dg(x)}{dx}=1+2x+3x^2+4x^3+\ldots$$

so $\frac{x}{(1-x)^2}$ is the gen. fun. for the sequence $0, 1, 2, 3, 4, \ldots$

Repeating this technique we find that,

$$x\frac{d}{dx}\left[x\left(\frac{dg(x)}{dx}\right)\right]=\frac{x(1+x)}{(1-x)^3}=x+2^2x^2+3^2x^3+4^2x^4+\ldots$$

so $\frac{x(1+x)}{(1-x)^3}$ generates $0^2, 1^2, 2^2, 3^2, \ldots.$

Apply summation operator to this, we get,

$$\frac{x(1+x)}{(1-x)^3}.\frac{1}{(1-x)}=\frac{x(1+x)}{(1-x)^4}$$

this genearates $0^2, 0^2+1^2, 0^2+1^2+2^2, 0^2+1^2+2^2+3^2, \ldots$

Hence co-eff of x^n in $\frac{x(1+x)}{(1-x)^4}$ is $\sum_{i=0}^{n} i^2$

But this co - eff can also be calcualted as,

$$\frac{x(1+x)}{(1-x)^4} = (x + x^2)(1-x)^{-4}$$

$$= (x + x^2)\left[\binom{-4}{0} + \binom{-4}{1}(-x) + \binom{-4}{2}(-x)^2 + \ldots\right]$$

so the co - eff of x^n is,

$$\binom{-4}{n-1}(-1)^{n-1} + \binom{-4}{n-2}(-1)^{n-2}$$

$$= (-1)^{n-1}\binom{4+(n-1)-1}{n-1}(-1)^{n-1} + (-1)^{n-2}\binom{4+(n-2)-1}{n-2}(-1)^{n-2}$$

$$= \binom{n+2}{n-1} + \binom{n+2}{n-2}$$

$$= \frac{(n+2)!}{3!(n-1)!} + \frac{(n+1)!}{3!(n-2)!}$$

$$= \frac{1}{6}[(n+2)(n+1)(n) + (n+1)(n)(n-1)]$$

$$= \frac{1}{6}(n)(n+1)[(n+2) + (n-1)]$$

$$= \frac{n(n+1)(2n+1)}{6}$$

Example: Find a formula for the sum of first n natural numbers using the generating function for the sequence 0, 1, 3, 6, 10, 15, ….

We know that,

$$\frac{1}{(1-x)^n} = \sum_{i=0}^{\infty}\binom{n+i-1}{i}x^i$$

for n = 3, we have, $\frac{1}{(1-x)^3} = \sum_{i=0}^{\infty}\binom{i+2}{i}x^i$

Thus the function $\frac{1}{(1-x)^3}$ genrates 1,3,6,10,15,...

Then $\frac{x}{(1-x)^3}$ generates 0,1,3,6,10,15,....

Now,

$$\sum_{k=0}^{n} k = \text{co-eff of } x^{n} \text{ in } \frac{x}{(1-x)^{3}}$$

$$= \text{co-eff of } x^{n} \text{ in } (1-x)^{-3}$$

$$= \text{co-eff of } x^{n-1} \text{ in } (1-x)^{-3}$$

$$= \binom{-3}{n-1}(-1)^{n-1} = (-1)^{n-1}\binom{3+(n-1)-1}{n-1}(-1)^{n-1}$$

$$= \binom{n+3}{n-1} = \frac{1}{2}(n)(n+1)$$

Summaries (*m* objects, *n* containers)

Objects Are Distinct	Containers Are Distinct	Some Containers May Be Empty	Number of Distributions
Yes	Yes	Yes	n^{m}
Yes	Yes	No	$n!S(m,n)$
Yes	No	Yes	$S(m,1)+S(m,2)+...+S(m,n)$
Yes	No	No	$S(m,n)$
No	Yes	Yes	$\binom{n+m-1}{m}$
No	Yes	No	$\binom{n+(m-n)-1}{m-n}$
No	No	Yes	(1) $p(m)$, for $n=m$
No	No	No	(2) $p(m,1)+p(m,2)+...+p(m,n)$, $n<m$ $p(m,n)$

$p(m.n)$:number of partitions of m into exactly n summands