Fundamental Principles of Counting

1.1 The Rules of Sum and Product

Our study of discrete and combinatorial mathematics beings with two basic principles of counting: the rules of sum and product. The statements and initial applications of these rules appear quite simple. In analyzing more complicated problems, one is often able to break down such problems into parts that can be solved using these basic Principles. We want to develop the ability to "decompose" such problems and piece together our partial solutions in order to arrive at the final answer. A good way to do this is to analyze and solve many diverse enumeration problems, Taking note of the principles being used. This is the approach we shall follow here.

Our first principle of counting can be stated as follows:

The Rule of Sum:

If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any of $m + n$ ways.

Note that when we say that a particular occurrence, such as a first task, can come about in m ways, these m ways are assumed to be distinct, unless a statement is made to the contrary. This will be true throughout the entire text.

Example 1.1

A College library has 40 textbooks on sociology and 50 textbooks dealing with anthropology. By the rule of sum, a student at this college can select among 40 + 50 = 90 textbooks in order to learn more about one or the other of these two subjects.

Example 1.2

The rule can be extended beyond two tasks as long as no pair of tasks can occur simultaneously. For instance, a computer science instructor who has, say, seven different introductory books each on C++, Java and Perl can recommend any one of these 21 books to a student who is interested in learning a first programming language.

Example 1.3

The computer science instructor of Example 1.2 has two colleagues. One of three colleagues has three textbooks on the analysis of algorithms, and the other has five such textbooks. If ***n*** denotes the maximum number of different books on this topic that this instructor can borrow from them, then $5 \leq n \leq 8$, for here both colleagues may own copies of the same textbook(s).

Example 1.4

Suppose a university representative is to be chosen either from 200 teaching or 300 non-teaching employees, and then there are 200 + 300 = 500 possible ways to choose this representative.

Extension of Sum Rule:

If tasks $T_1, T_2, \ldots\ldots, T_m$ can be done in $n_1, n_2, \ldots\ldots, n_m$ ways respectively and no two of these tasks can be performed at the same time, then the number of ways to do ***one*** of these tasks is $n_1 + n_2 + \ldots. + n_m$.

Example 1.5

If a student can chose a project either 20 from mathematics or 35 from computer science or 15 from engineering, then the student can choose a project 20 + 35 + 15 = 70 ways.

The following example introduces our second principle of counting.

Example 1.6

In trying to reach a decision on plant expansion, an administrator assigns 12 of her employees to two committees. Committee A consists of five members and is to investigate possible favorable results from such an expansion. The other seven employees, committee B, will scrutinize possible unfavorable repercussions. Should the administrator decide to speak to just one committee member before making her decision, then by the rule of sum there are 12 employees she can call upon for input. However, to be a bit more unbiased, she decides to speak with a member of committee B on Tuesday, before reaching a decision. Using the following principle, we find that she can select two such employees to speak with in 5 X 7 = 35 ways.

The rule of Product:

If a procedure can be broken down into first and second stages, and if there are ***m*** possible outcomes for the first stage and if, for each of these outcomes, there are ***n*** possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in ***mn*** ways.

Example 1.7

The drama club of Central University is holding tryouts for a spring play. With six men and eight women auditioning for the leading male and female roles, by the rule of product the director can cast his leading couple in 6 **X** 8 = 48 ways.

Example 1.8

Here various extensions of the rule are illustrated by considering the manufacture of license plates consisting of two letters followed by four digits.

a) If no letter or digit can be repeated, there are 26 **X** 25 **X** 10 **X** 9 **X** 8 **X** 7 = 3,276,000 different possible plates.

b) With repetitions of letters and digits allowed, 26 **X** 26 **X** 10 **X** 10 **X** 10 **X** 10 = 6,760,000 different license plates are possible.

c) If repetitions are allowed, as in part (**b**), how many of the plates have only vowels (A, E, I, O, U) and even digits? (0 is an even integer)

Example 1.9

In order to store data, a computer's main memory contains a large collection of circuits, each of which is capable of storing a bit — that is, one of the binary digits 0 or 1. These storage circuits are arranged in units called (memory) cells. To identify the cells in a computer's main memory, each is assigned a unique name called its address. For some computer's, such as embedded microcontrollers (as found in the ignition system for an automobile), an address is represented by an ordered list of eight bits, collectively referred to as a ***byte***. Using the rule of product, there are 2 x 2 x 2 x 2 x 2 x 2 x 2 x 2 = 2^8 = 256 such bytes. So we have 256 addresses that may be used for cells where certain information may be stored.
A kitchen appliance, such as a microwave oven, incorporates an embedded microcontroller. These "small computers" (such as the PICmicro microcontroller) contain

thousands of memory cells and use two-byte addresses to identify these cells in their main memory. Such addresses are made up of two consecutive bytes, or 16 consecutive bits. Thus there are 256 X 256 = 2^8 X 2^8 = 2^{16} = 65,536 available address that could be used to identifying cells in main memory. Other computers use addressing systems of four bytes. This 32-bit architecture is presently used in the Pentium processor, where there are as many as 2^8 X 2^8 X 2^8 X 2^8 = 232 = 4,294,967,296 addresses for use in identifying the cells in main memory. When a programmer deals with the UltraSPARC or Itanium processors, he or she considers memory cells with eight-byte addresses. Each of these addresses comprises 8 X 8 = 64 bits, and there are 2^{64} = 18,446,744,073,709,551,616 possible addresses for this architecture. (Of course, not all of these possibilities are actually used.)

Example 1.10

At times it is necessary to combine several different counting principles in the Solution of one problem. Here we find that the rules of both sum and product are needed to attain the answer.

At the AWL Corporation Mrs. Foster operates the Quick Snack Coffee Shop. The menu at her shop is limited: six kinds of muffins, eight kinds of sandwiches, and five beverages (hot coffee, hot tea, cola, and orange juice). Ms. Dodd, an editor at AWL, sends her assistant Carl to the shop to get her lunch — either a muffin and a hot beverage or a sandwich and a cold beverage.

By the rule of product, there are 6 X 2 = 12 ways in which Carl can purchase a muffin and hot beverage. A second application of this rule shows that there are 8 X 3 = 24 possibilities for a sandwich and cold beverage. So by the rule of sum, there are 12 + 24 = 36 ways in which Carl can purchase Ms. Dodd's lunch.

Example 1.11

A tourist can travel from Hyderabad to Tirupati in four ways (by plane, train, bus or taxi). He can then travel from Tirupati to Tirumala hills in five ways (by RTC bus, taxi, rope way, motorcycle or walk). Then the tourist can travel from Hyderabad to Tirumala hills in 4 X 5 = 20 ways.

Extension of Product Rule: Suppose a procedure consists of performing tasks $T_1, T_2, \ldots, T_m$ in order. Suppose task Ti can be performed in ***ni*** ways after the tasks $T_1, T_2, \ldots, T_{i-1}$ are performed, then the number of ways the procedure can be executed in the designated order is ***n1 , n2 , n3 , . . . ,*** n_m

Example 1.12

“Charmas” brand shirt available in 12 colors has a male and female version. It comes in four sizes for each sex, comes in three makes of economy, standard and luxury. Then the numbers of different types of shirts produced are 12 X 2 X 4 X 3 = 288.

Example 1.13

If there are 18 boys and 12 girls in a class, there are 18 + 12 = 30 ways of selecting 1 student (either a boy or a girl) as class representative.

Example 1.14

Suppose ***E*** is the event of selecting a prime number less than 10 and ***F*** is the event of selecting an even number less than 10. then ***E*** can happen in 4 ways. But, because 2 is an even prime, ***E*** and ***F*** can happen in only 4 + 4 – 1 = 7 ways.

Example 1.15

A bookshelf holds 6 different English books, 8 different French books, and 10 different German books. There are (***i***) (8) (9) (10) = 480 ways of selecting 3 books, 1 in each language; (***ii***) 6 + 8 + 10 = 24 ways of selecting 1 book in any one of languages.

Example 1.16

The scenario is as in Example 1.15. An English book and a French book can be selected in (6) (8) = 48 ways; an English book and a German book, in (6) (10) = 60 ways; a French book and a German book, in (8) (10) = 80 ways. Thus there are 48 + 60 + 80 = 188 ways of selecting 2 books in 2 languages.

Example 1.17

If each of the 8 questions in a multiple-choice examination has 3 answers (1 correct and 2 wrong), the number of ways of answering all questions is 38 = 6561.

Example 1.18

There are P(***6, 6***) = 720 6-letter "words" that can be made from the letters of word NUMBER, and there are P(6, 4) = 6!/2! = 360 4-letter "words". An unordered selection of ***r*** out of the ***n*** elements of ***X*** is called an ***r-combination*** of ***X***. In other words, any subset of ***X*** with r elements is an ***r***-combination of ***X***. The number of ***r***-combinations or ***r***-subsets of a set of n distinct objects is denoted by ***C (n, r)*** *("n choose r").* For each ***r***-subset of ***X*** there is unique complementary (***n – r***)-subset, whence the important relation ***C (n, r) = C (n, n – r).*** To evaluate ***C (n, r)***, note that an ***r-****permutation* of an ***n***-set ***X*** is necessarily a permutation of some ***r***-subset of ***X***. Moreover distinct ***r***-subsets generate distinct ***r-***permutations. Hence, by the sum rule,

$$P(n, r) = P(r, r) + P(r, r) + \ldots + P(r, r)$$

The number of terms on the right is the number of ***r-***subset of ***X***; i.e. ***C (n, r).*** Thus
P (***n, r***)-subset, whence the important relation ***C (n, r) = C (n, n – r).***

Example 1.19

From a class consisting of 12 computer science majors, 10 mathematics majors, and 9 statistics majors, a committee of 4 computer science majors, 4 mathematics majors, and 3 statistics majors is to be formed. There are

$$C(12,4) = \frac{12!}{4!8!} = \frac{12.11.10.9}{4.3.2.1} = 11.5.9 = 495$$

Ways of choosing 4 computer science majors, C(10, 4) = 210 ways of choosing 4 mathematics majors, and C(9, 3)+ 84 ways of choosing 3 statistics majors. By the product rule, the number of ways of forming a committee is thus (495)(210)(84) = 8,731,800.

Example 1.20

Refer to Example 1.18 in how many ways can a committee consisting of 6 or 9 members be formed such that all 3 majors are equally represented?

A committee of 6(with 2 from each group) can be formed in ***C(12,2).C(10,2).C(9,2)*** = 106,920 ways. The number of ways of forming a committee of 9 (with 3 from each group) is ***C(12,3).C(10,3).C(9,3)*** = 2,217,600. Then, by the sum rule the number of ways of forming a committee is 106,920 + 2,217,600 = 2,324,520.

Example 1.21

There are 15 married couples in a party. Find the number of ways of choosing a woman and a man from the party such that the two are ***(a)*** married to each other, ***(b)*** not married to each other.

(a) A woman can be chosen in 15 ways. Once a woman is chosen, her husband automatically chosen. So the number of ways of choosing a married couple is 15.
(b) A woman can be chosen in 15 ways. Among the 15 men in the party, one is her husband. Out of the 14 other men, one can be chosen in 14 ways. The product rule the gives (15)(14) = 210 ways.

Example 1.22

Find the number of ***(a)*** 2-digit even numbers, ***(b)*** 2-digit odd numbers, ***(c)*** 2-digit odd numbers with distinct digits, and ***(d)*** 2-digit even numbers with distinct digits.

Let ***E*** be the event of choosing a digit for the units' position, and ***F*** be the event choosing a digit for the tens' position.

(a) E can be done in 5 ways; ***F*** can be done in 9 ways. The number of ways of doing ***F*** does not depend upon how ***E*** is done; hence, the sequence {***E, F***} can be done in (5)(9) = 45 ways.

(b) The argument is as in ***(a)***: there are 45 2-digit odd numbers.

(c) If ***F*** is done first, the number of ways of doing ***E*** depends upon how ***F*** was done; so we cannot apply the product rule to the sequence {***F, E***}. But we can apply the product rule to the sequence {***E, F***}. There are 5 choices for the units' digit, and for each of these there are 8 choices for the tens' digit. So the sequence {***E, F***} can be done in 40 ways; i.e., there are 40 2-digit odd numbers with distinct digits.

(d) We distinguish two cases. If the units' digit is 0-which can be accomplished in 1 way-the tens' digit can be chosen in 9 ways. If 2,4,6, or 8 is chosen as units' digit, the tens' digit can be chosen in 8 ways. Thus the sum and product rules give a total of (1)(9)+(4)(8) = 41 ways.

Example 1.23

A computer password consists of a letter of the alphabet followed by 3 or 4 digits. Find

(a) the total number of passwords that can be created, and ***(b)*** the number of Passwords in which no digit repeats.

(a) The number of 4-charcter passwords is (26)(10)(10)(10), and the number of 5-charcter passwords is (26)(10)(10)(10)(10), by the product rule. So the total number of passwords is 26,000 + 260,000 = 286,000, by the sum rule.

(b) The number of 4-charcter passwords is (26)(10)(9)(8) = 18,720, the number of 5-charcter passwords is (26)(10)(9)(8)(7) = 131,040, for a total of 149,760.

Example 1.24

How many among the first 100,000 positive integers contain exactly one 3, one 4, and one 5 in their decimal representation?

It is clear that we may consider instead the 5-place numbers 00000 through 99999. The digit 3 can be in any one of the 5 places. Subsequently the digit 4 can be in any one of the remaining places. Then the digit 5 can be in one of 3 places. There are 2 places left, either of which may be filled by 7 digits. Thus there are (5)(4)(3)(7)(7) = 2940 integers in the desired category.

Example 1.25

Find the number of 3-digit even numbers with no repeated digits.

By problem 1.21***(d)***, the hundreds' and units' positions can be simultaneously filled in 41 ways. For each of these ways, the tens' position can be filled in 8 ways. Hence the desired number is (41)(8) = 328ways.

Example 1.26

A palindrome is a finite sequence of characters that reads the same forwards and backwards [GNUDUNG]. Find the numbers of 7-digit and 8-digit palindromes, under the restriction that no digit may appear more than twice.

By the mirror-symmetry of a palindrome (of length n), only the first $\lfloor (n+1)/2 \rfloor$ Positions need be considered. In our case this number is 4 for both lengths. Since the first digit may not be 0, there are 9 ways to fill the first position. There are then 10-1 = 9 ways to fill the second position; 10-2 = 8 ways for the third; 10-3 = 7 ways for the fourth. Thus there are (9)(9)(8)(7) = 4536 palindromic numbers of either length.

Example 1.27

In a binary palindrome the first digits is 1 and each succeeding digit may be 0 or 1. Count the binary palindromes of length n.

See problem 1.25. Here we have $\lfloor (n+1)/2 \rfloor - 1 = \lfloor (n-1)/2 \rfloor$ free positions, so the desired number is

Example 1.28

Find the number of proper divisors of 441,000. (A proper divisor of positive integer n is any divisor other than 1 and n)

Any integer can be uniquely expressed as product of powers of prime numbers; thus, $441{,}000 = (2^3)(3^2)(5^3)(7^2)$. Any divisor, proper or improper, of given number must be of the form $(2^a)(3^b)(5^c)(7^d)$, where $0 \le a \le 3$, $0 \le b \le 2$, $0 \le c \le 3$, and $0 \le d \le 2$. in this paradigm the exponent a can be chosen in 4 ways; b in 3 ways; c in 4 ways; d in 3 ways. So, by the product rule, the total number of proper divisors will be (4)(3)(4)(3) – 2 = 142.

Example 1.29

In a binary sequence every element is 0 or 1. Let X be the set of all binary sequences of length n**.** A switching function (Boolean function) of n variables is A function from X to the set Y = {0, 1}. Find the number of distinct switching functions of n variables**.**

The cardinality of X is $r = 2^n$. So the number of switching functions is 2^r.

1.2 Permutations

Continuing to examine applications of rule of product, we turn now to counting linear arrangements of objects. These arrangements are often called ***permutations*** when the objects are distinct. We shall develop some systematic methods for dealing with linear arrangements, starting with a typical example.

Example 1.14

In class of 10 students, five are to be chosen and seated in a row for a picture. How many such linear arrangements are possible?

The key word here is arrangement, which designates the importance of order. If A, B, C, . . ., I, J denote the 10 students, then BCEFI, CEFIB, and ABCFG are there such different arrangements, even though the first two involve the same five students.

To answer this question, we consider the positions and possible numbers of students we can choose in order to fill each position. The filling of position is a stage of our procedure.

10	X	9	X	8	X	7	X	6
1st position		**2nd position**		**3rd position**		**4th position**		**5th position**

Each of the 10 students can occupy the 1st position in the row. Because repetitions are not possible here, we can select only one of the nine remaining students to fill the 2^{nd} position. Continuing in this way, we find only six students to select from in order to fill the 5^{th} and final position. This yields a total of 30,240 possible arrangements of five students selected from the class of 10.

Exactly the same answer is obtained if the positions are filled from right to left namely, 6 X 7 X 8 X 9 X 10. if the 3^{rd} position is filled first, the 1st position second, the 4th osition third, the 5^{th} position fourth, and the 2^{nd} position fifth then answer is 9 X 6 X 10 X 8 X 7, still the same value, 30,240.

Definition 1.1

As in Example 1.14, the product of certain consecutive positive integers often comes into play in enumeration problems. Consequently, the following notation proves to be quite useful when we are dealing with such counting problems. It will frequently allow us to express our answers in a more convenient form.

For an integer $n \geq 0$, ***n*** factorial (denoted ***n!***) is defined by

$$0! = 1$$
$$n! = (n)(n-1)(n-2)\ldots(3)(2)(1), \text{ for } n \geq 1,$$

One finds that 1! = 1, 2! = 2, 3! = 6, 4! = 24, and 5! = 120, in addition, for each $n \geq 0$, (n + 1)! = (n + 1) (n!).

Before we proceed any further, let us try to get a somewhat better appreciation for how fast n! grows. We can calculate that 10! = 3,628,800, and it just so happens that this is exactly the number of seconds in six weeks, Consequently, 11! Exceeds the number of seconds in one year, 12! Exceeds the number in 12 years, and 13! Surpasses the number of seconds in century.

If we make use of the factorial notation, the answer in Example 1.14 can be Expressed in the following more compact form:

$$10\times9\times8\times7\times6 = 10\times9\times8\times7\times6\times\frac{5\times4\times3\times2\times1}{5\times4\times3\times2\times1} = \frac{10!}{5!}$$

Definition 1.2

Given a collection of n distinct objects. Any (linear) arrangement of these objects is called a permutation of the collection.

Starting with the letters a, b, c, there are six ways to arrange, or permute, all of the letters: abc, acb, bac, bca, cab, cba. If we are interested in arranging only two of the letters at a time, there are six such size – 2 permutations: ab, ba, ac, ca, bc, cb.

If there are ***n*** distinct objects and r is an integer, with $1 \leq r \leq n$, then by the rule of product, the number of permutations of size r for the n objects are

P(n, r) =	n	X	(n - 1)	X	(n - 2)	X . . . X	(n - r + 1)
	1st position		**2nd position**		**3rd position**		**rth position**

$$= (n)(n-1)(n-2)...(n-r+1)\times\frac{(n-r)(n-r-1)...(1)(2)(3)}{(n-r)(n-r-1)...(1)(2)(3)}$$

$$= \frac{n!}{(n-r)!}$$

For r 0, P (n, 0) = 1 = n!/ (n- 0)!, so P (n, r) = n!/(n-r)! holds for all $0 \leq r \leq n$. A special case of this result is Example 1.14, where n = 10, r = 5, and P (10, 5) = 30, 240. When permuting all of the n objects in the collection, we have r = n and find that P (n, n) = n!/0!=n!.

Note, for example, that if $n \geq 2$, then P (n, 2) = n!/(n-2)! = n(n-1). When n>3 one finds that P(n,n-3) = n!/[n-(n-3)]!=n!/3!=(n)(n-1)(n_2)…(5)(4).

The number of permutations of size r, where $0 \leq r \leq n$, from a collection of n objects, is P (n, r) =n!/(n-r)! (Remember that P (n, r) counts (linear) arrangements in which the objects cannot be repeated.) However, if repetitions are allowed, then by the rule of product there are nr possible arrangements, with $r \geq 0$.

Example 1.15

The number of words of three distinct letters formed from the letters of word "JNTU" is P (4, 3) = 4!/(4 - 3)!=24. If repetitions are allowed, the number of possible six – letter sequence is 46 = 4096.

Example 1.16

In how many ways can eight men and eight women be seated in a row if (**a**) any person may sit next to any other (**b**) men and women must occupy alternate seats (**c**) generalize this result for n men and n women.

Here eight men and eight women are 16 indistinguishable objects.
a) The number of permutations 16 chosen form 16 objects is P (16, 16) = 16!
= 20922789890000.
b) Here men and women are distinct (different)

i)

M	W	M	W	M	W	M	W	M	W	M	W	M	W	M	W
8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1

Man sitting first: the number of ways is 8! 8!

ii)

W	M	W	M	W	M	W	M	W	M	W	M	W	M	W	M
8	8	7	7	6	6	5	5	4	4	3	3	2	2	1	1

Woman sitting first: 8! 8!
Thus the number of ways men and women occupy
Alternatively is 8! 8! + 8! 8! = $2(8!)^2$

c) Any person may sit: (2n)!
Men and women sit alternatively: $2(n!)^2$

Example 1.17

A committee of eight is to be formed from 16 men and 10 women. In how many ways can the committee be formed if (**a**) there are no restrictions (**b**) there must be 4 men and 4 women (**c**) there should be an even number of women (**d**) more women than men (**e**) at least 6 men.

a) No distinction between men and women. Problem is to choose 8 out of a set of 26 persons. So the number of ways 8 are chosen out of 26 is C(26, 8) = 26!/ 89! (18!) = 2.480721325 x 1017

b) First stage choose 4 men out of 16 given by C(16, 4). Second stage choose 4 women out of 10 in C(10, 4)ways. Using product rule, the number of ways in which the committee consisting of 4 men and women is C(16,4) C(10,4) = 1,820 x 210 = 382,200.

c) If 2*i* even number of women are chosen, then the remaining 8 -2*i* members of the committee should be men. By product rule, C(10, 2*i*)C(16, 8-2*i*). Then the total number of ways is

$$\sum_{i=1}^{4} \binom{10}{2i}\binom{16}{8-2i}$$

d) Since the strength of the committee is 8, there should be 5 or more women so that women outnumber men. Using product rule, the number of ways is.

$$\sum_{i=5}^{8} \binom{10}{i} \binom{16}{8-i}$$

e) When the number of men is 6 or more we get by a similar argument, the number of ways as

$$\sum_{i=5}^{8} \binom{16}{i} \binom{10}{8-i}$$

Example 1.18

The number of permutations of the letters in the word COMPUTER is 8!. If only five of the letters are used, the number of permutations (of size 5) is P(8, 5) = 8!/(8-5)! = 8!/3! = 6720. If repetitions of letters are allowed, the number of possible 12-letter sequences is $8^{12} = 6.872 \times 10^{10}$.

A	**B**	**L**	**L**	**A**	**B**	**L1**	**L2**	**A**	**B**	**L2**	**L1**
A	**L**	**B**	**L**	**A**	**L1**	**B**	**L2**	**A**	**L2**	**B**	**L1**
A	**L**	**L**	**B**	**A**	**L1**	**L2**	**B**	**A**	**L2**	**L1**	**B**
B	**A**	**L**	**L**	**B**	**A**	**L1**	**L2**	**B**	**A**	**L2**	**L1**
B	**L**	**A**	**L**	**B**	**L1**	**A**	**L2**	**B**	**L2**	**A**	**L1**
B	**L**	**L**	**A**	**B**	**L1**	**L2**	**A**	**B**	**L2**	**L1**	**A**
L	**A**	**L**	**L**	**L1**	**A**	**B**	**L2**	**L2**	**A**	**B**	**L1**
L	**A**	**A**	**B**	**L1**	**A**	**L2**	**B**	**L2**	**A**	**L1**	**B**
L	**B**	**L**	**L**	**L1**	**B**	**A**	**L2**	**L2**	**B**	**A**	**L1**
L	**B**	**A**	**A**	**L1**	**B**	**L2**	**A**	**L2**	**B**	**L1**	**A**
L	**L**	**A**	**B**	**L1**	**L2**	**A**	**B**	**L2**	**L1**	**A**	**B**
L	**L**	**B**	**A**	**L1**	**L2**	**B**	**A**	**L2**	**L1**	**B**	**A**

(a) **(b)**

Example 1.19

Unlike example 1.18, the number of (linear) arrangements of the four letters in BALL is 12, not 4! (= 24), the reason is that we do not have four distinct letters to arrange. To get the 12 arrangements, we can list them as in table 1.1(a).

If the two L's are distinguished as L_1, L_2, then we can use our previous ideas on permutations of distinct objects; with the four distinct symbols B, A, L_1, L_2, we have 4!=24 permutations. These are listed in Table 1.1(b). Table 1.1 reveals that for each arrangement in which the L's are indistinguishable there corresponds a pair of permutations with distinct L's. Consequently,

2 X (Number of arrangements of the letters B, A, L, L)
= (Number of permutations of the symbols B, A, L1, L2),

And the answer to the original problem of finding all the arrangements of the four letters in BALL is 4!/2 = 12.

Example 1.20

Using the idea developed in Example 1.19, we now consider the arrangements of all nine letters in DATABASES.

There are 3! = 6 arrangements with the A's distinguished for each arrangements in which the A's are not distinguished. For example, DA_1 TA_2 BA_3 SES, $DA_1TA_3BA_2SES$, $DA_2TA_1BA_3SES$, $DA_2TA_3BA_1SES$, $DA_3TA_1BA_2SES$, and $DA_3TA_2BA_1SES$ all correspond to DATABASES, when we remove the subscripts on A's. In addition, to the arrangement $DA_1TA_2BA_3SES$ there corresponds the pair of permutations $A_1TA_2BA_3S_1ES_2$ and $DA_1TA_2BA_3S_2ES_1$, when the S's are distinguished. Consequently,

(2!)(3!) (Number of arrangements of the letters in DATABASES) =
(Number of permutations of the symbols D, A_1, T, A_2, B, A_3, S_1, E, S_2)
So the number of arrangements of the nine letters in DATABASES is 9!/(2!3!)
= 30,240.

Before stating a general principle for arrangements with repeated symbols, note that in our prior two examples we solved a new type of problem by relating it to previous enumeration principles. This practice is common in mathematics in general, and often occurs in the derivations of discrete and combinational formulas.

If there are n objects with n_1 indistinguishable objects of an r^{th} type, where $n_1 + \dots + n_r = n$, then there are $\frac{n!}{n_1!n_2!\dots n_r!}$ (linear) arrangements of the given n objects

Example 1.21

The MASSASAUGA is a brown and white venomous snake indigenous to North America. Arranging all of the letters in MASSASAUGA. We find that there are

$$\frac{10!}{4!3!1!1!1!} = 25,200$$

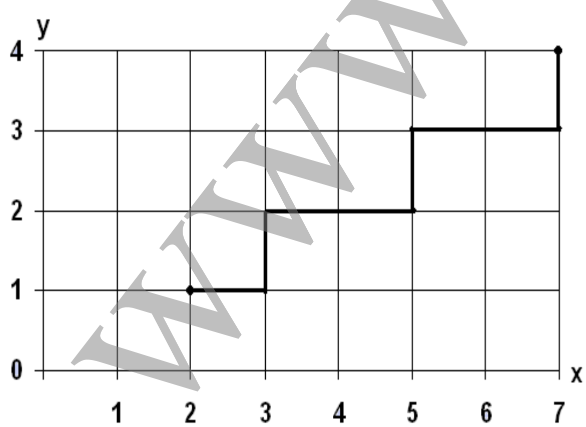
Possible arrangements. Among these are

$$\frac{7!}{3!1!1!1!1!} = 840$$

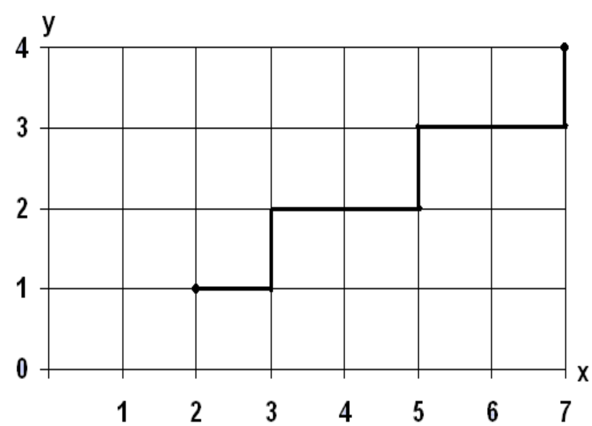
In which all four A's are together. To get this last result, we considered all arrangements of the seven symbols AAAA (one symbol), S, S, S, M, U, G.

Example 1.22

Determine the number of (staircase) paths in the ***xy***-plane from (2, 1) to (7, 4), Where each such path is made up of individual steps going one unit to the right (R) or one unit upward (U). The blue lines in Fig. 1.1 show two of these Paths.



(a) U,R,R,R,U,U,R,R



(a) U,R,R,R,U,U,R,R

Figure 1.1

Beneath each path in Fig. 1.1 we have listed the individual steps. For example, in part (a) the list R, U, R, R, U, R, R, U indicates that starting at the point (2, 1), we first move one unit to the right [to (3, 1)], then one unit upward [to (3, 2)], followed by two units to the right [to (5, 2)], and so on, until we reach the point (7, 4). The path consists of five R's for moves to the right and three U's for moves upward.

The path in part (b) of the figure is also made up of five R's and three U's. in general, the overall trip from (2, 1) to (7, 4) requires 7 – 2 = 5 horizontal moves to the right and 4 – 1 = 3 vertical moves upward. Consequently, each path corresponds to a list of five R's and U's, and the solution for the number of paths emerges as the number of arrangements of the five R's and three U's, which is 8!/(5! 3!) = 56.

Example 1.23

We now do something a bit more abstract and prove that if ***n*** and ***k*** are positive integers with ***n*** = 2***k***, then n!/2k is an integer. Because our argument relies on Counting, it is an example of a ***combinatorial proof***.

Consider the ***n*** symbols ***x1, x1, x2 , x2***……………………., ***xk , xk***. The number of ways in which we can arrange all of these ***n*** =2***k*** symbols is an integer that equals

$$\frac{n!}{\underbrace{2!2!\ldots 2!}_{k \text{ factors of } 2!}} = \frac{n!}{2^k}$$

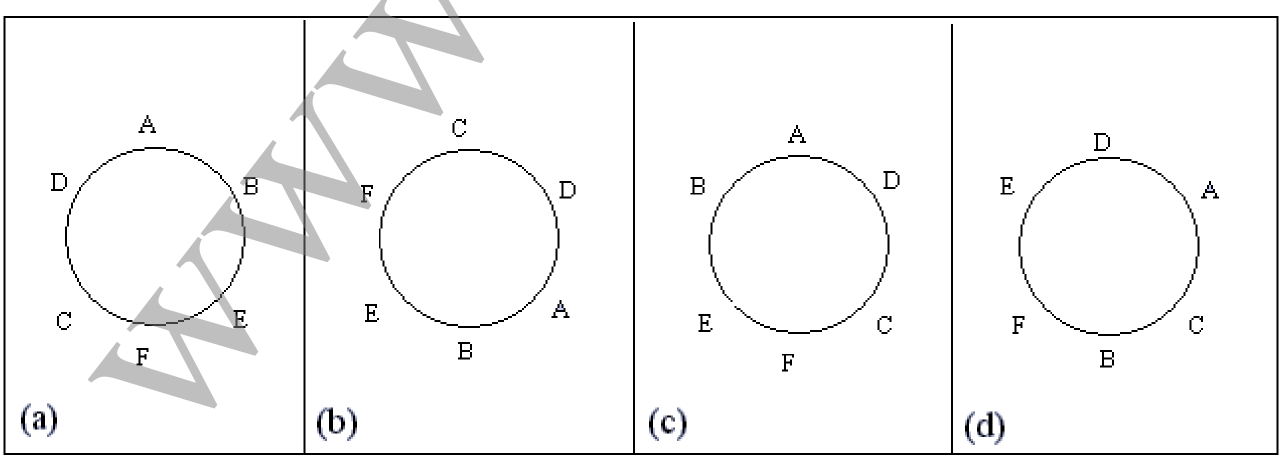


Figure 1.2

We shall try to relate this problem to previous ones we have already encountered. Consider Figs. 1.2 (a) and (b). Starting at the top of the circle and moving clockwise, we list the distinct linear arrangements ABEFCD and CDABEF, which correspond to the same circular arrangements. In addition to these two, four other linear arrangements – BEFCDA, DABEFC, EFCDAB, and FCDABE — are found to correspond to the same circular arrangements as in (a) or (b). So inasmuch as each circular arrangement corresponds to six linear arrangements,

We have 6 X (Number of circular arrangements of A, B, . . ., F) =
(Number of linear arrangements of A, B……F) = 6!.

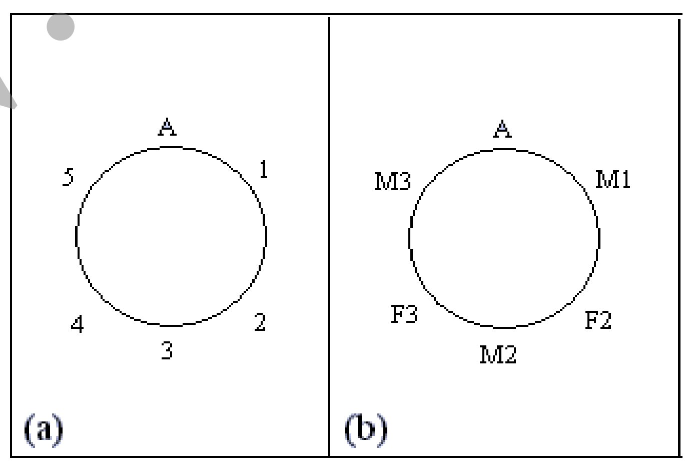
Consequently, There are 6!/6 = 5! = 120 arrangements of A, B,….F around the circular table.

Example 1.25

Suppose now that the six people of Example 1.24 are three married couples and that A, B, and C are the females. We want to arrange the six people around the table so that the sexes alternate. (Once again, arrangements are considered identical if one can be obtained from the other by rotation.)

Before we solve this problem, let us solve Example 1.24 by an alternative method, which will assist us in solving our present problem. If we place A at the table as shown in Fig. 1.3(a), five locations (clockwise from A) remain to be filled. Using B, C,…,F to fill.

Figure 1.3



These five positions is the problem of permuting B, C, . . ., F in a linear manner, and this be done in 5! = 120 ways.

To solve the new problem of alternating the sexes, consider the method shown in Fig. 1.3(b). A (a female) is placed as before. The next position, clockwise from A, is marked M1 (Male 1) and can be filled in three ways. Continuing clockwise from A, position F2 (Female 2) can be filled in two ways. Proceeding in this Manner, by the rule of product, there are 3 x 2 x 2 x 1 x 1 = 12 ways in which these six people can be arranged with no two men or women seated next to each other.

1.3 Combinations: The Binomial Theorem

The standard Deck of playing Cards Consists of 52 cards comprising four suits: Clubs, diamond, hearts, and spades. Each suit has 13 cards: ace, 2, 3,…, 9, 10, jack, queen, king. If we are asked to draw three cards from a standard deck, in succession and without replacement, then by the rule of product there are

$$52 \times 51 \times 50 = \frac{52!}{49!} = P(52,3)$$

possibilities, one of which is AH (ace of hearts), 9C (nine clubs), KD (King of diamonds). If instead we simply select three cards at one time from the deck so that the order of selection of the cards is no longer AH-9C-KD, AH-KD-9C, 9C-AH-KD,9C-KD-AH, KD-9C-AH, and KD-AH-9C all correspond to just one (unordered) selection. Consequently, each selection, or combination, of three cards, with no reference to order, corresponds to 3! Permutations of three cards. In equation form this translates into

(3!) x (Number of selection of size 3 from a deck of 52)
= Number of permutations of size 3 for the 52 cards

Consequently, three cards can be drawn, without replacement, from a standard deck in 52!/(3! 49!) = 22,100 ways.

If we start with n distinct objects, each selection, or combination, of r of these objects, with no reference to order, corresponds to r! Permutations of size r from the n objects. Thus the number of combinations of size r from a collection of size n is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}, 0 \le r \le n.$$

In addition to C (n, r) the symbol $\binom{n}{r}$ is frequently used. Both C (n, r) and $\binom{n}{r}$ are sometimes read “n choose r.” Note that for all n ≥ 0, C (n, r) = C (n, n) = 1. Further, for all n ≥ 1, C (n, 1) = C (n, n-1) = n. when 0 ≤ n < r, then C (n, r) = $\binom{n}{r} = 0$

A word to the wise! When dealing with any counting problem, we should ask ourselves about the importance of order in the problem, when order is relevant, we think in terms of permutations and arrangements and the rule of product. When order is not relevant, combinations could play a key role in solving the problem.

Example 1.26

A hostess is having a dinner party for some members of her charity committee. Because of the size of her home, she can invite only 11 of the 20 committee members. Order is not important, so she can invite "the lucky 11" in C (20, 11) = $\binom{20}{11}$ = 20!/(11! 9!) = 167, 960 ways. However, once the 11 arrive, how she arranges them around her rectangular dinning table is an arrangement problem. Unfortunately, no part of theory of combinations and permutations can help our hostess deal with "the offended nine" who were not invited.

Example 1.27

Lynn an Patti decide to buy a PowerBall ticket. To win the grand prize for PowerBall one must match five numbers selected from 1 to 49 inclusive and then must also match the powerball, an integer from 1 to 42 inclusive. Lynn selects the five numbers (between 1 and 49 inclusive). This she can do in $\binom{49}{5}$ ways (since matching does not involve order). Meanwhile Patti selects the powerball – here there are $\binom{42}{1}$ possibilities. Consequently, by the rule of product, Lynn and Patti can select the six numbers for their PowerBall ticket in $\binom{49}{5}\binom{42}{1}$ =80,089,128 ways.

Example 1.28

a) A student taking a history examination is directed to answer any seven of 10 essay questions. There is no concern about order here, so the student can answer the examination in $\binom{10}{7} = \frac{10!}{7!3!} = \frac{10\times9\times8}{3\times2\times1} = 12 ways$

b) If the student must answer three questions from the first five and four questions from the last five, three questions can be selected from the first five in $\binom{5}{3}$ = 10 ways, and the other four questions can be selected in $\binom{5}{4}$ = 5 ways. Hence, by the rule of product, the student can complete the examination in $\binom{5}{3}\binom{5}{4}$ = 10 X 5 = 50 ways.

c) Finally, should the directions on this examination indicate that the student must answer seven of the 10 questions where at least three are selected from the first five, then there are three cases to consider:

i) The student answers three of the first five questions and four of five: by the rule of product this can happen in $\binom{5}{3}\binom{5}{4}$ = 10 X 5 = 50 ways, as in part (b).

ii) Four of the first five questions and three of the last five questions are selected by the student: this can come about in $\binom{5}{4}\binom{5}{3}$ = 5 X 10 = 50 ways – again by the rule of product.

iii) The student decides to answer all five of the first five questions and two of the last five: The rule of product tells us that last $\binom{5}{5}\binom{5}{2}$ case can occur in = 1 X 10 = 10 ways.

$$\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2}$$

Combining the results for cases (i), (ii), and (iii), by the rule of sum we find that the student can make $\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2}$ = 50 + 50 + 10 = 110 selections of seven (out of 10) questions where each selection includes at least three of the first five questions.

Example 1.29

a) At Rydell High School, the gym teacher must select nine girls from the junior and senior classes for a volleyball team. If there are 28 juniors and 25 seniors, she can make the selection in $\binom{53}{9}$ = 4,431,613,550 ways.

b) If two juniors and one senior are the best spikers and must be on the team, then the rest of the team can be chosen in $\binom{50}{6} = \binom{25}{5}$15,890,700 ways.

c) For a certain tournament that team must comprise four juniors and five seniors. The teacher can select the four juniors in $\binom{28}{4}$ ways. For each of these selections she has ways to choose the five seniors. Consequently, by the rule of product, she can select her team in $\binom{28}{4}\binom{25}{5}$ = 1,087,836,750 ways for this particular tournament.

Some problems can be treated from the viewpoint of either arrangements or combinations, depending on how one analyzes the situation. The following Example demonstrates this.

Example 1.30

The gym teacher of Example 1.29 must make up four volleyball teams of nine girls each from the 36 freshman girls in her P.E. class. In how many ways can she select these four terms? Call the teams A, B, C, and D.

a) To form team A, she can select any nine girls from the 36 enrolled $\binom{36}{9}$ in ways. For team B the selection process yields $\binom{27}{9}$ possibilities. This leaves $\binom{18}{9}$ and $\binom{9}{9}$ possible ways to select teams C and D, respectively. So by the rule of product, the four teams can be chosen in

$$\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9}=\left(\frac{36!}{9!27!}\right)\left(\frac{27!}{9!18!}\right)\left(\frac{18!}{9!9!}\right)\left(\frac{9!}{9!0!}\right)=\frac{36!}{9!9!9!9!}=2.145\times10^{19}\,ways$$

b) For an alternative solution, consider the 36 students lined up as follows:

$$\begin{matrix} 1st & 2nd & 3rd & & 35th & 36th \\ student & student & student & \cdots & student & student \end{matrix}$$

To select the four teams, we must distribute nine A's, nine B's, nine C's and nine D's in the 36 spaces. The number of ways in which this can be done is the number of arrangements of 36 letters comprising nine each of A, B, C, and D. This is now the familiar problem of arrangements of nondistinct objects, and the answer is

$$\frac{36!}{9!9!9!9!},\ as\ in\ part\ (a)$$

Our next example points out how some problems require the concepts of both arrangements and combinations for their solutions.

Example 1.31

The number of arrangements of the letters in TALLAHASSEE is

$$\frac{11!}{3!2!2!2!1!1!} = 831{,}600\ .$$

How many of these arrangements have no adjacent A's?

When we disregard the A's, there are

$$\frac{8!}{2!2!2!1!1!} = 5040$$

Ways to arrange the remaining letters. One of these 5040 ways is shown in the following figure, where the arrows indicate nine possible locations for the three A's.

$$\uparrow E \uparrow E \uparrow S \uparrow T \uparrow L \uparrow L \uparrow S \uparrow H \uparrow$$

Three of these locations can be selected in $\binom{9}{3}$ = 84 ways, and because this is also possible for all the other 5039 arrangements of E, E, S, T, L, L, S, H, by the rule of product there are 5040 X 84 = 423,360 arrangements of the letters in TALLAHASSEE with no consecutive A's.

Before proceeding we need to introduce a concise way of writing the sum of list of $\boldsymbol{n}$ + 1 terms like a_{m}, a_{m+1}, a_{m+2}, . . ., a_{m+n}, where $\boldsymbol{m}$ and $\boldsymbol{n}$ are integers and $\boldsymbol{n} \geq 0$. This notation is called the Sigma Notation because it involves the capital Greek letter Σ; we use it to represent a summation by writing

$$a_m + a_{m+1} + a_{m+2} + \ldots + a_{m+n} = \sum_{i=m}^{m+n} a_i .$$

Here, the letter $\boldsymbol{i}$ is called the index of the summation, and this index accounts for all integers starting with the *lower limit* $\boldsymbol{m}$ and Continuing on up to (and including) the *upper limit* $\boldsymbol{m + n}$.

We may use this following notation

$$1)\sum_{i=3}^{7} a_i = a_3 + a_4 + a_5 + a_5 + a_6 + a_7 = \sum_{j=3}^{7} a_j \quad \text{for there is}$$

nothing special about the letter ***i***.

$$2)\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30 = \sum_{k=0}^{4} k^2, because\, 0^2 = 0.$$

$$3)\sum_{i=11}^{100} i^3 = 11^3 + 12^3 + 13^3 + ... + 100^3 = \sum_{j=12}^{101} (j-1)^3 = \sum_{k=10}^{99} (k+1)^3$$

$$4)\sum_{i=7}^{10} 2i = 2(7) + 2(8) + 2(9) + 2(10) = 68 = 2(34)$$

$$5)\sum_{i=3}^{3} a_i = a_3 = \sum_{i=4}^{4} a_{i-1} = \sum_{i=2}^{2} a_{i+1}$$

$$6)\sum_{i=1}^{5} a = a + a + a + a + a = 5a$$

Furthermore, using this summation notation, we see that one can express the answer to part (c) of Example 1.28 as

$$\binom{5}{3}\binom{5}{4} + \binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2} = \sum_{i=3}^{5}\binom{5}{i}\binom{5}{7-i} = \sum_{j=2}^{4}\binom{5}{7-j}\binom{5}{j}$$

We shall find use for this new notation in the following example and in many other places throughout the remainder of this book

Example 1.32

In the studies of algebraic coding theory and the theory of computer languages, we consider certain arrangements, called ***strings***, made up from a prescribed ***alphabet*** of symbols. If the prescribed alphabet consists of the symbols 0, 1, and 2, for example, then 01, 11, 21, 12, and 20 are five of the nine strings of length 2. Among the 27 strings of length 3 are 000,012, 202, and 110.

In general, if n is any positive integer, then by the rule of product there are $3''$ strings of length n for the alphabet 0,1, and 2. If $x = x_1x_2x_3 \ldots x_n$ is one of these strings, we define the weight of x, denoted wt(x), by wt(x) = $x_1 + x_2 + x_3 + \ldots + x_n$. For example, wt(12) = 3 and wt(22) = 4 for the case where n = 2; wt(10l) = 2, wt(210) = 3, and wt(222) = 6 for $n = 3$.

Among the 3^{10} strings of length 10, we wish to determine how many have even weight. Such a string has even weight precisely when the number of 1's in the string is even.

There are six different cases to consider. If the string x contains no 1's, then each of the 10 locations in x can be filled with either 0 or 2, and by the rule of product there are 210 such strings. When the string contains two 1's, the locations for these two 1's can be selected in $\binom{10}{2}$ ways. Once these two locations have been specified, there are 2^8 ways to place either 0 or 2 in the other eight positions. Hence there are $\binom{10}{2}$ 2^8 strings of even weight that contain two 1's. The numbers of strings for the other four cases are given in Table1.2.

Consequently, by the rule of sum, the number of strings of length 10 that have even weight is

$$2^{10} + \binom{10}{2}2^8 + \binom{10}{4}2^6 + \binom{10}{6}2^4 + \binom{10}{8}2^2 + \binom{10}{10} = \sum_{n=0}^{5}\binom{10}{2n}2^{10-2n}$$

Number of 1's	Number of Strings	Number of 1's	Number of Strings
4	$\binom{10}{4}2^6$	8	$\binom{10}{8}2^2$
6	$\binom{10}{6}2^4$	10	$\binom{10}{10}$

Table 1.2

Often we must be careful of *overcounting*—a situation that seems to arise in what may appear to be rather easy enumeration problems. The next example demonstrates how overcounting may come about.

Example 1.33

a) Suppose that Ellen draws five cards from a standard deck of 52 cards. In how many ways can her selection result in a hand with no clubs? Here we are interested in counting all five-card selections such as

i) Ace of hearts, three of spades, four of spades, six of diamonds, and the jack of diamonds.

ii) Five of spades, seven of spades, ten of spades, seven of diamonds, and me king of diamonds.

iii) Two of diamonds, three of diamonds, six of diamonds, ten of diamonds, and the jack of diamonds.

If we examine this more closely we see that Ellen is restricted to selecting her five cards from the 39 cards in me deck that are not clubs. Consequently, she can make her selection in $\binom{39}{5}$ ways.

b) Now suppose we want to count the number of Ellen's five-card selections that contain at least one club. These are precisely the selections that were not counted in part (a). And since there are $\binom{52}{5}$ possible five-card hands in total, we find that

$$\binom{52}{5} - \binom{39}{5} = 2,598,960 - 575,757 = 2,023,203$$

of all five-card hands contain at least one club.

c) Can we obtain the result in part (b) in another way? For example, since Ellen wants to have at least one club in the five-card hand, let her first select a club. This she can do in $\binom{13}{1}$ ways. And now she doesn't care what comes up for the other four cards. So after she eliminates the one club chosen from her standard deck, she can then select the other four cards in $\binom{51}{4}$ ways. Therefore, by the rule of product, we count the number of selections here as

$$\binom{13}{1}\binom{51}{4} = 13 \text{ X } 249,900 = 3,248,700$$

Something here is definitely wrong! This answer is larger than that in part (b) by more than one million hands. Did we make a mistake in part (b)? Or is something wrong with our present reasoning?

For example, suppose that Ellen first selects
the three of clubs
and then selects
the five of clubs,
king of clubs,
seven of hearts, and
jack of spades.

If, however, she first selects
the five of clubs
and then selects
the three of clubs,
king of clubs,
seven of hearts, and
jack of spades,
is her selection here really different from the prior selection we mentioned? Unfortunately, no! And the case where she first
selects.

the king of clubs
and then follows this by selecting
the three of clubs,
five of clubs,
seven of hearts, and
jack of spades
is not different from the other two selections mentioned earlier.
Consequently, this approach is wrong because we are overcounting — by considering like selections as if they were distinct.

d) But is there any other way to arrive at the answer in part (b)? Yes! Since the five-card hands must each contain at least one club, there are five cases to consider. These are given in Table 1.3. From the results in Table 1.3 we see, for example, that there are $\binom{13}{2}\binom{39}{5}$ five-card hands that contain exactly two clubs. If we are interested in having exactly three clubs in the hand, then the results in the table indicate that there are $\binom{13}{3}\binom{39}{2}$ such hands.

Since no two of the cases in Table 1.3 have any five-card hand in common, the number of hands that Ellen can select with at least one club is

$$\binom{13}{1}\binom{39}{4}+\binom{13}{2}\binom{39}{3}+\binom{13}{3}\binom{39}{2}+\binom{13}{4}\binom{39}{1}+\binom{13}{5}\binom{39}{0}$$

$$=\sum_{i=1}^{5}\binom{13}{i}\binom{39}{5-i}$$

$$=(13)(82{,}251)+(78)(9139)+(286)(741)+(715)(39)+(1287)(1)$$

$$=2{,}023{,}203$$

Table 1.3

Number of clubs	Number of Ways to Select This Number of Clubs	Number of Cards That Are Not Clubs	Number of Ways to Select This Number of Non clubs
1	$\binom{13}{1}$	4	$\binom{39}{4}$
2	$\binom{13}{2}$	3	$\binom{39}{3}$
3	$\binom{13}{3}$	2	$\binom{39}{2}$
4	$\binom{13}{4}$	1	$\binom{39}{1}$
5	$\binom{13}{5}$	0	$\binom{39}{0}$

We shall close this section with three results related to the concept of combinations.

First we note that for integers ***n, r,*** with ***n ≥ r ≥0,*** $\binom{n}{r} = \binom{n}{n-r}$. This can be established algebraically from the formula for $\binom{n}{r}$, but we prefer to observe that when dealing with a selection of size ***r*** from a collection of ***n*** distinct objects, the selection process leaves behind ***n – r*** objects. Consequently, $\binom{n}{r} = \binom{n}{n-r}$ affirms the existence of a correspondence between the selections of size ***r*** (objects chosen) and the selections of size ***n - r*** (objects left behind). An example of this correspondence is shown in Table 1.4, where ***n*** = 5, ***r*** = 2, and the distinct objects are 1, 2, 3, 4, and 5.

This type of correspondence will be more formally defined in Chapter 5 and used in other counting situations.
Our second result is a theorem from our past experience in algebra.

Theorem 1.1
The Binomial Theorem. If ***x*** and ***y*** are variables and ***n*** is a positive integer, then

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \ldots$$
$$+ \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0 = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$$

Before considering the general proof, we examine a special case. If ***n*** = 4, the coefficient of ***x2y2*** in the expansion of the product

(x+y)	***(x+y)***	***(x+y)***	***(x+y)***
1st factor	2nd factor	3rd factor	4th factor

is the number of ways in which we can select two ***x***'s from the four ***x***'s, one of which is available in each factor. (Although the ***x***'s are the same in appearance, we distinguish them as the ***x*** in the first factor, the ***x*** in the second factor, ... , and the ***x*** in the fourth factor.

Also, we note that when we select two ***x***'s, we use two factors, leaving us with two other factors from which we can select the two ***y***'s that are needed.) For example, among the possibilities, we can select (1) ***x*** from the first two factors and ***y*** from the last two or (2) ***X*** from the first and third factors and ***y*** from the second and fourth. Table 1.5 summarizes the six possible selections.

Consequently, the coefficient of x^2y^2 in the expansion of $(x + y)^4$ is $\binom{4}{2} = 6$, the number of ways to select two distinct objects from a collection of four distinct objects.

Table 1.4

Selections of Size r = 2 (Objects Chosen)		Selections of Size n - r = 3 (Objects Left Behind)	
1. 1,2	6. 2,4	1. 3,4,5	6. 1,3,5
2. 1,3	7. 2,5	2. 2,4,5	7. 1,3,4
3. 1,4	8. 3,4	3. 2,3,5	8. 1,2,5
4. 1,5	9. 3,5	4. 2,3,4	9. 1,2,4
5. 2,3	10. 4,5	5. 1,4,5	10. 1,2,3

Table 1.5

Factors Selected for x	Factors Selected for y
1. 1,2	1. 3,4
2. 1,3	2. 2,4
3. 1,4	3. 2,3
4. 2,3	4. 1,4
5. 2,4	5. 1,3
6. 2,5	6. 1,2

Now we turn to the proof of the general case.

Proof: In the expansion of the product

(x+y)	(x+y)	(x+y)	(x+y)
1st Factor	**2nd Factor**	**3rd Factor**	**4th Factor**

The coefficient of $x^k y^{n-k}$**, where** $0 \leq k \leq n$**,** is the number of different ways in which we can select **k** ***x*'s** [and consequently **(*n* - *k*) *y*'s]** from the *n* available factors. (One way, for example, is to choose $\boldsymbol{x}$ from the first $\boldsymbol{k}$ factors and $\boldsymbol{y}$ from the last $\boldsymbol{n-k}$ factors) The total number of such selections of size $\boldsymbol{k}$ form a collection of size $\boldsymbol{n}$ is C $(\boldsymbol{n}, \boldsymbol{k}) = \binom{n}{k}$, and from this the binomial theorem follows.

Example 1.34

In view of this theorem, is often referred to as a binomial coefficient. Notice that it is also possible to express the result of Theorem 1.1 as

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{n-k} x^k y^{n-k}.$$

a) From the binomial theorem it follows that the coefficient of $\boldsymbol{x^5 y^2}$ in the expansion of $(x + y)^7$ is $\binom{7}{5} = \binom{7}{2} = 21$

b) To obtain the coefficient of $\boldsymbol{a^5 b^2}$ in the expansion of $(2a - 3b)^7$, replace 2$\boldsymbol{a}$ by $\boldsymbol{x}$ and 3$\boldsymbol{b}$ by $\boldsymbol{y}$. From the binomial theorem the coefficient of $\boldsymbol{x^5 y^2}$ in $(x + y)^7$ is and

$$\binom{7}{5} x^5 y^2 = \binom{7}{5} (2a)^5 (-3b)^2 = \binom{7}{5} (2)^5 (-3)^2 a^5 b^2 = 6048 a^5 b^2.$$

Corollary 1.1

For each integer n > 0,

$$a) \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n, and$$

$$b) \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n \binom{n}{n} = 0.$$

Proof: Part (a) follows from the binomial theorem when we set x = y = 1. When x = -1 and y = 1, part (b) results.

Our third and final result generalizes the binomial theorem and is called the *multinomial theorem*.

Theorem 1.2

For positive integers ***n, t,*** the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots x_t^{n_t}$ in the expansion of $(x^1 + x^2 + x^3 + \ldots + x^t)^n$ is

$$\frac{n!}{n_1!\, n_2!\, n_3! \ldots n_t!}$$

Where each n_i is an integer with $0 \le n_i \le n$, for all $1 \le i \le t$, and $n_1 + n_2 + n_3 + \ldots + n_t = n$.

Proof: As in the proof of the binomial theorem, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \ldots x_t^{n_t}$ is the number of ways we can select x_1 from n_1 of the n factors, x_2 from n_2 of the $n - n_1$ remaining factors, x_3 from n_3 of the $n - n_1 - n_2$ now remaining factors, …., and x_t from nt of the last $n - n_1 - n_2 - n_3 - \ldots - n_{t-1} = n_t$ remaining factors. This can be carried out, as in part (a) of Example 1.30, in

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-n_3-\ldots-n_{t-1}}{n_t}$$

ways. We leave to the reader the details of showing that this product is equal to $\dfrac{n!}{n_1!n_2!n_3!\ldots n_t!}$,

which is also written as $\dbinom{n}{n_1, n_2, n_3, \ldots, n_t}$

and is called a *multinomial coefficient*. (When t = 2 this reduces to a binomial coefficient)

Example 1.35

a) In the expansion of $(x + y + z)^7$ it follows from the multinomial theorem that the coefficient of $x^2y^2z^3$ is $\binom{7}{2,2,3} = \left(\frac{7!}{2!2!3!}\right) = 210$, while the coefficient of xyz^5 is $\binom{7}{1,1,5} = 42$ and that x^3z^4 is $\binom{7}{3,0,4} = \left(\frac{7!}{3!0!4!}\right) = 35$.

b) Suppose we need to know the coefficient of ***a2b3c2d5*** in the expansion of ***(a + 2b – 3c + 2d + 5)16***. If we replace ***a*** by ***v***, ***2b*** by ***w***, - ***3c*** by ***x***, 2d by ***y***, and 5 by ***z***, then we can apply the multinomial Theorem to $(v + w + x + y + z)^{16}$ and determine the coefficient of $v^2w^3x^2y^5z^4$ as $\binom{16}{2,3,2,5,4}$ = 302,702,400. But

$$\binom{16}{2,3,2,5,4}(a)^2(2b)^3(-3c)^2(2d)^5(5)^4$$

$$=\binom{16}{2,3,2,5,4}(1)^2(2)^3(-3)^2(2)^5(5)^4\left(a^2b^3c^2d^5\right)$$

$$=435{,}891{,}456{,}000{,}000a^2b^3c^2d^5$$

1.4 Combinations with Repetition

When repetitions are allowed, we have seen that for ***n*** distinct objects an arrangement of size ***r*** of these objects can be obtained in n^r ways, for an integer $r \geq 0$. We now turn to the comparable problem for combinations and once again obtain a related problem whose solution follows from our previous enumeration principles.

Example 1.36

On their way home from track practice, seven high school freshmen stop at a restaurant, where each of them has one of the following: a cheeseburger, a hot dog, a taco, or a fish sandwich. How many different purchases are possible (from the viewpoint of the restaurant)?

Let c, h, t, and f represent cheeseburger, hot dog, taco, and fish sandwich, respectively. Here we are concerned with how many of each item are purchased, not with the order in which they are purchased, so the problem is one of selections, or combinations, with repetition.

In Table 1.6 we list some possible purchases in column (a) and another means of representing each purchase in column (b).

Table 1.6

<table>
<tr><td>1. c, c, h, h, t, t, f
2. c, c, c, c, h, t, f
3. c, c, c, c, c, c, f
4. h, t, t, f, f, f, f
5. t, t, t, t, t, f, f
6. t, t, t, t, t, t, t
7. f, f, f, f, f, f, f</td><td>8. x x | x x | x x | x
9. x x x x | x | x | x
10.x x x x x x | | | x
11.| x | x x | x x x x
12.| | x x x x x | x x
13.| | x x x x x x x |
14.| | | x x x x x x x</td></tr>
<tr><td>(a)</td><td>(b)</td></tr>
</table>

For a purchase in column (b) of Table 1.6 we realize that each x to the left of the first bar (|) represents a c, each x between the first and second bars represents an h, the x's between the second and third bars stand for t’s, and each x to the right of the third bar stands for an f. The third purchase, for example, has three consecutive bars because no one bought a hot dog or taco; the bar at the start of the fourth purchase indicates that there were no cheeseburgers in that purchase.

Once again a correspondence has been established between two collections of objects, where we know how to count the number in one collection. For the representations in column (b) of Table 1.6, we are enumerating all arrangements of 10 symbols consisting of seven x's and three l's, so by our correspondence the number of different purchases for column (a) is.

$$\frac{10\;!}{7\,!3\,!} = \binom{10}{7}$$

In this example we note that the seven x's (one for each freshman) correspond to the size of the selection and that the three bars are needed to separate the 3+1= 4 possible food items that can be chosen.

When we wish to select, with repetition, ***r*** of ***n*** distinct objects, we find (as in Table 1.6) that we are considering all arrangements of ***r*** x's and ***n*** - 1 l's and that their number is

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}$$

Consequently, the number of combinations of n objects taken r at a time, *with repetition*, is C (***n + r – 1, r***).
(In Example 1.36, ***n*** = 4, ***r*** = 7, so it is possible for ***r*** to exceed ***n*** when repetitions are allowed)

Example 1.37

A donut shop offers 20 kinds of donuts. Assuming that there are at least a dozen of each kind when we enter the shop, we can select a dozen donuts in C (20 + 12 - 1, 12) = C(31, 12) = 141,120,525 ways. (Here ***n*** = 20, ***r*** = 12.)

Example 1.38

President Helen has four vice presidents: (1) Betty, (2) Goldie, (3) Mary Lou, and (4) Mona. She wishes to distribute among them $1000 in Christmas bonus checks, where each check will be written for a multiple of $100.

a) Allowing the situation in which one or more of the vice presidents get nothing, President Helen is making a selection of size 10 (one for each unit of $100) from a collection of size 4 (four vice presidents), with repetition. This can be done in C(4 + 10 - l, 10) = C(13, 10) = 286 ways.

b) If there are to be no hard feelings, each vice president should receive at least $ 100. With this restriction, President Helen is now faced with making a selection of size 6 (the remaining six units of $100) from the same collection of size 4, and the choices now number C(4+6- 1,6) = C(9, 6) = 84. [For example, here the selection 2, 3, 3,4, 4, 4 is interpreted as follows: Betty does not get anything extra—for there is no 1 in the selection. The one 2 in the selection indicates that Goldie gets an additional $100. Mary Lou receives an additional $200 ($100 for each of the two 3's in the selection). Due to the three 4's, Mona's bonus check will total $100 + 3($100) = $400.]

c) If, each vice president must get at least $100 and Mona, as executive vice president, gets at least $500, then the number of ways President Helen can distribute the bonus checks is

$$\underbrace{C(3+2-1,2)}_{\substack{Mona \quad gets \\ exactly \ \$500}}+\underbrace{C(3+1-1,1)}_{\substack{Mona \quad gets \\ exactly \ \$600}}+\underbrace{C(3+0-1,0)}_{\substack{Mona \quad gets \\ exactly \ \$700}}=10=\underbrace{C(4+2-1,2)}_{\substack{Using \quad the \\ technique\ in \quad part(b)}}$$

Having worked examples utilizing combinations with repetition, we now consider two examples involving other counting principles as well.

Example 1.39

In how many ways can we distribute seven bananas and six oranges among four children so that each child receives at least one banana?

After giving each child one banana, consider the number of ways the remaining three bananas can be distributed among these four children. Table 1.7 shows four of the distributions we are considering here. For example, the second distribution in part (a) of Table 1.7—namely, 1, 3, 3—indicates that we have given the first child (designated by 1) one additional banana and the third child (designated by 3) two additional bananas. The corresponding arrangement in part (b) of Table 1.7 represents this distribution in terms of three b's and three bars.

These six symbols—three of one type (the b's) and three others of a second type (the bars)—can be arranged in 6!/(3! 3!) = C(6, 3) = C(4+3 - 1, 3) =20 ways. [Here n = 4, r = 3.] Consequently, there are 20 ways in which we can distribute the three additional bananas among these four children. Table 1.8 provides the comparable situation for distributing the six oranges. In this case we are arranging nine symbols—six of one type (the o's) and three of a second type (the bars). So now we learn that the number of ways we can distribute the six oranges among these four children is 9!/(6! 3!) =

C(9, 6) = C(4+6 - 1, 6) = 84 ways. [Here n = 4, r = 6.] Therefore, by the rule of product, there are 20 X 84 = 1680 ways to distribute the fruit under the stated conditions.

Table 1.7

(a)	(b)
1. 1, 2, 3	**5. b | b | b |**
2. 1, 3, 3	**6. b | | b b |**
3. 3, 4, 4	**7. | | b | b b**
4. **4, 4, 4**	**8. | | | b b b**

Table 1.8

(a)	(b)
1. 1, 2, 2, 3, 3, 4	**5. 0 | 0 0 | 0 0 | 0**
2. 1, 2, 2, 4, 4, 4	**6. 0 | 0 0 | | 0 0 0**
3. 2, 2, 2, 3, 3, 3	**7. | 0 0 0 | 0 0 0 |**
4. 4, 4, 4, 4, 4, 4	**8. | | | 0 0 0 0 0 0**

Example 1.40

A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 (blank) spaces between the symbols, with at least three spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

There are 12! ways to arrange the 12 different symbols, and for each of these arrangements there are 11 positions between the 12 symbols. Because there must be at least three spaces between successive symbols, we use up 33 of the 45 spaces and must now locate the remaining 12 spaces. This is now a selection, with repetition, of size 12 (the spaces) from a collection of size 11 (the locations), and this can be accomplished in C (11 +12—1, 12) ==646,646 ways.

Consequently, by the rule of product the transmitter can send such messages with the required spacing in $(12!)\binom{22}{12}$ = 3.097 X 1014 ways.

In the next example an idea is introduced that appears to have more to do with number theory than with combinations or arrangements. Nonetheless, the solution of this example will turn out to be equivalent to counting combinations with repetitions.

Example 1.41

Determine all integer solutions to the equation

$x_1 + x_2 + x_3 + x_4 = 7$, where $x_i \geq 0$ for all $1 \leq i \leq 4$.

One solution of the equation is $x_1 = 3, x_2 = 3, x_3 = 0, x_4 = 1$. (This is different from a solution such as $x_1 = 1, x_2 = 0, x_3 = 3, x_4 = 3$, even though the same four integers are being used.) A possible interpretation for the solution $x_1 = 3, x_2 = 3, x_3 = 0, x_4 = 1$ is that we are distributing seven pennies (identical objects) among four children (distinct containers), and here we have given three pennies to each of the first two children, nothing to the third child, and the last penny to the fourth child. Continuing with this interpretation, we see that each nonnegative integer solution of the equation corresponds to a selection, with repetition, of size 7 (the identical pennies) from a collection of size 4 (the distinct children), so there are C(4 +7 - 1,7) = 120 solutions.

At this point it is crucial that we recognize the equivalence of me following:

a) The number of integer solutions of the equation

$$x_1 + x_2 + ... + x_n = r, \quad x_i \geq 0, \quad 1 \leq i \leq n.$$

b) The number of selections, with repetition, of size r from a collection of size n.

c) The number of ways r identical objects can be distributed among n distinct containers.

In terms of distributions, part (c) is valid only when the r objects being distributed are identical and the n containers are distinct. When both the r objects and the n containers are distinct, we can select any of the n containers for each one of the objects and get n^r distributions by the rule of product.

When the objects are distinct but the containers are identical, we shall solve the problem using the Stirling numbers of the second kind (Chapter 5). For the final case, in which both objects and containers are identical, the theory of partitions of integers (Chapter 9) will provide some necessary results.

Example 1.42

In how many ways can one distribute 10 (identical) white marbles among six distinct containers?

Solving this problem is equivalent to finding the number of nonnegative integer solutions to the equation $x_1 + x_2 + \ldots + x_6 = 10$. That number is the number of selections of size 10, with repetition, from a collection of size 6. Hence the answer is C(6 + 10 — 1, 10) = 3003.

We now examine two other examples related to the theme of this Section.

Our next two examples provide applications from the area of computer science. Furthermore, the second example will lead to an important summation formula that we shall use in many later chapters.

Example 1.43

Consider the following program segment, where i, j, and k are integer variables.

```
for i := 1 to 20 do
  for j : = 1 to i do
    for k : = 1 to j do
      print (i * j + k)
```

How many times is the **print** statement executed in this program segment?

Among the possible choices for i, j, and k (in the order i-first, j-second, k-third) that will lead to execution of the **print** statement, we list (1) 1, 1, 1; (2) 2, 1, 1; (3) 15, 10, 1; and (4) 15, 10, 7. We note that i = 10, j == 12, k = 5 is not one of the selections to be considered, because j = 12 > 10 = i; this violates the condition set forth in the second **for** loop. Each of the above four selections where the **print** statement is executed satisfies the condition $1 \le k \le j \le i \le 20$. In fact, any selection a, b, c ($a \le b \le c$) of size 3, with repetitions allowed, from the list 1, 2, 3,.... 20 results in one of the correct selections: here, k = a, j = b, i = c. Consequently the **print** statement is executed

$$\binom{20+3-1}{3} = \binom{22}{3} = 1540 \ \textit{times}$$

If there had been r (≥ 1) **for** loops instead of three, the **print** statement would have been executed $\binom{20+r-1}{r}$ times.

Example 1.44

Here we use a program segment to derive a summation formula. In this program segment, the variables ***i, j, n,*** and counter are integer variables. Furthermore, we assume that the value of ***n*** has been set prior to this segment.

```
counter : = 0
 for i : = 1 to n do
   for j : = 1 to i do
    counter : = counter +1
```

From the results in Example 1.43, after this segment is executed the value of (the variable) *counter* will be $\binom{n+2-1}{2} = \binom{n+1}{2}$.

(This is also the number of times that the statement
(*) counter : = counter +1
is executed.)

This result can also be obtained as follows: when i := 1, then j varies from 1 to 1 and (*) is executed once; when i is assigned the value 2, then j varies from 1 to 2 and (*) is executed twice; j varies from 1 to 3 when i is assigned the value 3, and (*) is executed three times; in general, for $1 \le k \le n$, when i := k, then j varies from 1 to k and (*) is executed k times. In total, the variable counter is incremented [and the statement (*) is executed] 1+2+3+…+n times.

Consequently,

$$\sum_{i=1}^{n} i = 1+2+3+...+n = \binom{n+1}{2} = \frac{n(n+1)}{2}$$

The derivation of this summation formula, obtained by counting the same result in two different ways, constitutes a combinatorial proof.