The Principle of Inclusion and Exclusion

8.1 The Principle of Inclusion and Exclusion

In this section we develop some notation for stating this new counting principle. Then we establish the principle by a combinatorial argument. Following this, a wide range of examples demonstrate how this principle may be applied.

We shall motivate the Principle of Inclusion and Exclusion with a series of three examples, the first two which will be reminiscent of the work we did with counting and Venn diagrams in section 3.3.

Example 8.1

Let S represent the set of 100 students enrolled in the freshman engineering program at Central College. Then $|S| = 100$. Now let c_1, c_2 denote the following conditions (or properties) satisfied by some of the elements of S:

c_1: A student at Central College is among the 100 students in the freshman engineering program and is enrolled in Freshman Composition.

c_2: A student at Central College is among the 100 students in the freshman engineering program and is enrolled in Introduction to Economics.

Suppose that 35 of these 100 students are enrolled in Freshman Composition and that 30 of them are enrolled in Introduction to Economics. We shall denote this by

$$N(c_1) = 35 \text{ and } N(c_2) = 30$$

If nine of these 100 students are enrolled in both Freshman Composition and Introduction to Economics then we write $N(c_1,c_2) = 9$.

Further, of these 100 students, there are 100 – 35 = 65 who are not taking Freshman Composition. Denoting |S| by N, we can designate this by writing $N(c_1) = N - N(c_1)$. In a similar way we designate that there are $N(c_2) = N - N(c_2) = 100 - 30 = 70$ of these students who are not taking Introduction to Economics. The number who are not taking Freshman Composition and who are not taking Introduction to Economics is $N(c_1 c_2) =$ $N(c_1) - N(c_1c_2) = 35 - 9 = 26$. Likewise, of these 100 students, there are $N(c_1 c_2) = N(c_2) - N(c_1,c_2) = 30 - 9 = 21$ who are enrolled in Introduction to Economics but not in Freshman

Composition. Of particular interest are those students (from among these 100 freshmen) who are taking Freshman Composition and they are also not taking Introduction to Economics.

Their number is $N(\bar{c}_1 \bar{c}_2)$. And since $N(\bar{c}_1) = N(\bar{c}_1 c_2) + N(\bar{c}_1 \bar{c}_2)$,
We learn that $N(\bar{c}_1 \bar{c}_2) = N(\bar{c}_1) - N(\bar{c}_1 c_2) = 65 – 21 = 44$.

The preceding observations also demonstrate that

$$N(\bar{c}_1 \bar{c}_2) = N(\bar{c}_1) - N(\bar{c}_1 c_2) = \left[N - N(\bar{c}_1) \right] - \left[N(\bar{c}_2) - N(c_1 c_2) \right]$$

$$= N - N(\bar{c}_1) - N(\bar{c}_2) + N(c_1 c_2) = N - \left[N(c_1) - N(c_2) \right] + N(c_1 c_2)$$

$$= 100 - [35 + 30] + 9 = 44, \quad \text{as we saw above}$$

From the Venn diagram in Fig. 8.1, we see that if $N(c_1)$ denotes the number of elements of S in the left-hand circle and $N(c_2)$ denotes the number in the right-hand circle, then $N(c_1 c_2)$ is the number of these elements from S in the overlap, while $N(\bar{c}_1 \bar{c}_2)$ counts those elements of S that are outside the union of these two circles. Consequently, we see once again – this time the figure – that

$$N(\bar{c}_1 \bar{c}_2) = N - [N(c_1) + N(c_2)] + N(c_1 c_2),$$

where the last term is added on because it was eliminated twice in the term $[N(c_1) + N(c_2)]$. (Also, at this point, the reader may wish to look back at the second formula following Example 3.26 to find the same result presented with a different notation.)

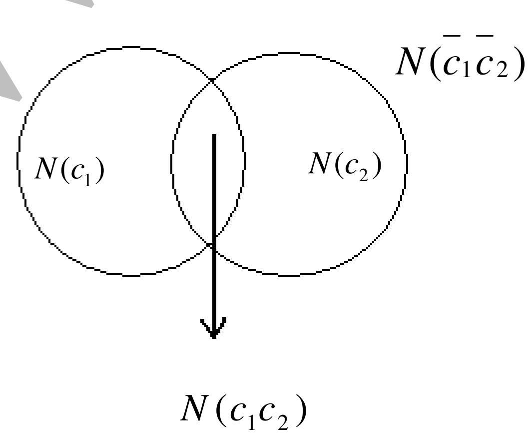


Figure 8.1

[Before we advance to our next example where we will introduce a third condition, let us note that $N(\bar{c}_1\bar{c}_2)$ is not the same as $N(\overline{c_1c_2})$. For $N(\overline{c_1c_2}) = N - N(c_1c_2) = 100 - 9 = 91$, in this example, while $N(\bar{c}_1\bar{c}_2) = 44$, as we learned earlier. However, $N(\bar{c}_1 \text{ or } \bar{c}_2) = N(\overline{c_1c_2}) = 91$ $= 65 + 70 - 44 = N(\bar{c}_1) + N(\bar{c}_2) - N(\bar{c}_1\bar{c}_2)$.]

Example 8.1

We start with the same 100 students as in Example 8.1 and the same conditions c_1, c_2, but now we consider a third condition, given as follows:

c_3: A student at Central College is among the 100 students in the freshman engineering program and is enrolled in Fundamentals of Computer Programming.

It is still the case that $N(c_1) = 35$, $N(c_2) = 30$, $N(c_1c_2) = 9$, but now we are also given that $N(c_3) = 30$, $N(c_1c_3) = 11$, $N(c_2c_3) = 10$, and $N(c_1c_2c_3) = 5$ (that is, there are five of these 100 freshmen who are taking Freshman Composition, Introduction to Economics, and Fundamentals of Computer Programming). Looking to Fig. 8.2, we learn that

$$N(\overline{c_1c_2c_3}) = N - [N(\bar{c}_1) + N(\bar{c}_2) + N(\bar{c}_3)] + [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3).$$

So here we have $N(\overline{c_1c_2c_3}) = 100 - [35+30+30] + [9+11+10] - 5 = 30$. That is, out of these 100 students there are 30 who are not enrolled in any of the courses: (i) Freshman Composition; (ii) Introduction to Economics; or (iii) Fundamentals of Computer Programming.

[We also learn here that $N(\bar{c}_3) = 70 = 100 - 30 = N - N(c_3)$, $N(\bar{c}_1\bar{c}_3)$,$= 46 = 100 - [35 + 30] + 11 = N - [N(c_1) + N(c_3)] + N(c_1c_3)$, and $N(\bar{c}_2\bar{c}_3) = 50 = 100 - [30 + 30] + 10 = N - [N(c_2) + N(c_3)] + N$ Furthermore, we note the similarity here with the result for $|\bar{A} \cap \bar{B} \cap \bar{C}|$ Given in the second formula following Example 3.27.]

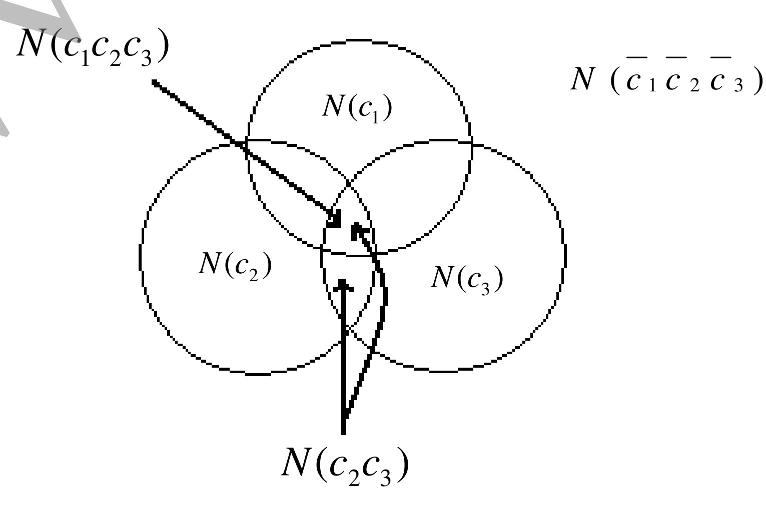


Figure 8.2

Example 8.2

Based on the results in the pervious two examples we may now feel that for given finite set S (with |S| = N) and four conditions c_1, c_2, c_3, c_4 we should have

$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = N - [N(c_1)+N(c_2)+N(c_3)+N(c_4)] \qquad (*)$$

$$+ [N(c_1c_2) + N(c_1c_3) + N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_3c_4)]$$

$$- [N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)]$$

$$+ N(c_1c_2c_3c_4).$$

To show that this is the case we consider an arbitrary elements x from S and show that it is counted the same number of times on both sides of the above equation.

0) If x satisfies only one of the conditions, then it is counted once on the left side of Eq. (*) [in $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4)$], and once on the right side of Eq. (*) [in N].

1) If satisfies only one of the conditions, say c1, then it is not counted at all on the left side of Eq. (*). But on the right side Eq. (*), x is continued once in N and once in $N(c_1)$, for a total of 1 – 1 = 0 times.

2) Now suppose that x satisfies conditions c2, c4 but does not satisfy conditions c1, c3. Once again x is not counted on the left hand side of Eq. (*). For the right side of Eq. (*), x is counted once in N, once in each of N(c2) and N(c4), and then once in N(c2c4), totaling $1 - [1+1] + 1 = 1 - \binom{2}{1} + \binom{2}{2} = 0$ times.

3) Continuing with the case for three conditions, we'll suppose here That x satisfies conditions c1,c2, and c4, but not c3. as in the previous two cases, x is not counted on the left side of Eq. (*). On the right side of Eq. (*), x is counted once in N, once each of N(c1), N(c2), and N(c4), once in each of N(c1c2), N(c1c4), and N(c2c4), and, finally, once in N (c1c2c4). So on the right side of Eq. (*), x is counted

$$1 - [1+1+1] + [1+1+1] - 1 = 1 - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} = 0 \text{ times, in total.}$$

4) Finally, if x satisfies all four of conditions c1, c2, c3, c4, then once again it is not counted on the left side of Eq. (*). On the right side of Eq. (*), x is counted once for each of the 16 terms on the right side of this equation – for a total of 1 – [1+1+1+1] + [1+1+1+1+1+1] – [1+1+1+1] + 1 = $1 - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + \binom{4}{4} = 0$ times.

Consequently, from these preceding five cases we have shown that the two sides of Eq. (*) count the same elements from S, and this provides a combinatorial proof for the formula for $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4)$.

So now we shall reconsider the situation in Example 8.2 and introduce a fourth condition as follows:

c_4: A student at central college is among the 100 students in the freshman engineering program & is enrolled in Introduction to Design.

We already know that $N(c_1) = 35$, $N(c_2) = 30$, $N(c_3) = 30$, $N(c_1c_2) = 9$, $N(c_1c_3) = 11$, $N(c_2c_3) = 10$, and $N(c_1c_2c_3) = 5$. If $N(c_4) = 41$, $N(c_1c_4) = 13$, $N(c_2c_4) = 14$, $N(c_3c_4) = 10$, $N(c_1c_2c_4) = 6$, $N(c_2c_3c_4) = 6$, and $N(c_1c_2c_3c_4) = 4$, then, using the equation we derived above, if follows that $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = 100 - [35+30+30+41] + [9+11+13+10+14+10] - [5+6+6+6] + 4 = 100 - 136 + 67 - 23 + 4 = 12$.

Thus, of the 100 students in the freshman engineering program at Central College, there are 12 who are not taking any of the four courses: Freshman Composition, Introduction to Economics, Fundamentals of Computer Programming, or Introduction to Design.

If we are interested in the number (from these 100 students) who are taking Freshman Composition, but none of the other three courses, then we should want to compute $N(c_1\bar{c}_2\bar{c}_3\bar{c}_4)$ To do so we start by observing that

$$N(\bar{c}_2\bar{c}_3\bar{c}_4) = N(c_1\bar{c}_2\bar{c}_3\bar{c}_4) + N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4),$$

Which can be established by an argument similar to the one above for $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4)$.

This then leads us to

$$N(c_1\bar{c}_2\bar{c}_3\bar{c}_4) = N(\bar{c}_2\bar{c}_3\bar{c}_4) - N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4),$$

Using the result in Example 8.2 we find that

$$N(\bar{c}_2\bar{c}_3\bar{c}_4) = N - [N(c_2) + N(c_3) + N(c_4)] + [N(c_2c_3) + N(c_2c_4) + N(c_3c_4)]$$

$$- N(c_2c_3c_4)$$

$$= 100 - [30 + 30 + 41] + [10 + 14 + 10] - 6 = 27, \textit{and}$$

$$N(c_1\bar{c}_2\bar{c}_3\bar{c}_4) = N(\bar{c}_2\bar{c}_3\bar{c}_4) - N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = 27 - 12 = 15.$$

So there are 15 students in this set of 100 who are taking Freshman Composition, but none of the other courses: Introduction to Economics, Fundamentals of Computer Programming, or Introduction to Design.

Further, we also observe that

$$N(c_1\bar{c}_2\bar{c}_3\bar{c}_4) = N(\bar{c}_2\bar{c}_3\bar{c}_4) - N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4)$$

$$\{N - [N(c_2)+N(c_3)+N(c_4)] + [N(c_2c_3)+N(c_2c_4)+N(c_3c_4)]$$

$$- N(c_2c_3c_4)\} - \{N - [N(c_1)+N(c_2)+N(c_3)+N(c_4)]$$

$$+ [N(c_1c_2)+N(c_1c_3)+N(c_1c_4)+N(c_2c_3)+N(c_2c_4)+N(c_3c_4)]$$

$$- [N(c_1c_2c_3)+N(c_1c_2c_4)+N(c_1c_3c_4)+N(c_2c_3c_4)] + N(c_1c_2c_3c_4)\}, or$$

$$N(c_1\bar{c}_2\bar{c}_3\bar{c}_4) = N(c_1) - [N(c_1c_2)+N(c_1c_3)+N(c_1c_4)] + [N(c_1c_2c_3)+N(c_1c_2c_4)+N(c_1c_3c_4)] + N(c_1c_2c_3c_4)$$

So here $N(c_1\bar{c}_2\bar{c}_3\bar{c}_4)$ = 35 – [9+11+13]+[5+6+6] – 4 = 35 – 33 + 17 – 4 = 15 as we found above.

Having seen the results in Examples 8.1, 8.2, 8.3, now it is time for us to generalize these results and establish the Principle of Inclusion and Exclusion. To do so we once again let S be set with |S| = N, and we let $c_1, c_2, \ldots, c_t$ be a collection of t conditions or properties – each of which may be satisfied by some of the elements in S. Some elements of S may satisfy more than one of the conditions, whereas others may not satisfy any of them. For all $1 \le i \le t$, $N(c_i)$ will denote the number of elements in S that satisfy condition c_i. (Elements of S are counted here when they satisfy only condition c_i, as well as when they satisfy c_i and other conditions c_j, for $j \ne i$.) for all i, j Є {1, 2, 3,. . . , t} where $i \ne j$, $N(c_ic_j)$ will denote the number of elements in S that satisfy only c_i, c_j, and perhaps some others. [$N(c_ic_j)$ does not count the elements in S that satisfy only c_ic_j.] Continuing, if $1 \le i, j, k \le t$ are three distinct integers, then $N(c_ic_jc_k)$ denotes the number of elements in S satisfying, perhaps among others, each of the conditions c_j, c_j, and c_k.

For each $1 \le i \le t$, N = N – N denotes the number of elements in S that do not satisfy conditions ci. If $1 \le i, j \le t$ with $i \ne j$, N = the number of elements in S that do not satisfy either of the conditions ci or cj. [This is not the same as N$(\overline{c_i c_j})$], as we observed At the end of Example 8.1.

With the necessary preliminaries now in hand we state the following theorem.

Theorem 8.1

The Principle of Inclusion and Exclusion. Consider a set S, with |S| = N, and conditions ci, $1 \le i \le t$, each of which may be satisfied by some of the elements of S. the number of elements of S that satisfy none of the conditions ci, $1 \le i \le t$, is denoted by $\overline{N} = N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\ldots\overline{c_t})$ where

$$\overline{N} = N - [N(c_1) + N(c_2) + N(c_3) + \ldots + N(c_t)] \qquad (1)$$

$$+ [N(c_1c_2) + N(c_1c_3) + \ldots + N(c_1c_t) + N(c_2c_3) + \ldots + N(c_{t-1}c_t)]$$

$$- [N(c_1c_2c_3) + N(c_1c_2c_4) + \ldots + N(c_1c_2c_t) + N(c_1c_3c_4) + \ldots$$

$$+ N(c_1c_2c_t) + \ldots + N(c_{t-2}c_{t-1}c_t) + \ldots + (-1)^t N(c_1c_2c_3\ldots c_t) \quad \text{or}$$

$$\overline{N} = N - \sum_{1\le i\le t} N(c_i) + \sum_{1\le i<j\le t} N(c_i\,c_j) - \sum_{1\le i<j\le k\le t} N(c_i\,c_j) + \ldots + (-1)^t N(c_1c_2c_3\ldots c_t). \qquad (2)$$

Proof: Although this result can be established by applying the Principle of Mathematical Induction to the number t of conditions, we shall give a combinatorial proof. The argument will be reminiscent of the ideas we saw in Example 8.3 in establishing the formula for $N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$.

For each x Є S we show that x contributes the same count, either 0 or 1, To each side of Eq. (2).

If x satisfies none of the conditions, then x is counted once in $\overline{N}$ and once in N, but not in any of the other terms in Eq. (2). Consequently, x contributes a count of 1 to each side of the equation.

The other possibility is that x satisfies exactly r of the conditions where $1 \le r \le t$. In this case x contributes nothing to $\overline{N}$. But on the right-hand side of Eq. (2), x is counted

1) One time in N.
2) times in $\sum_{1\le i\le t} N(c_i)$. (Once for each of the r conditions.)
3) $\binom{r}{2}$ times in $\sum_{1\le i<j\le t} N(c_i\,c_j)$. (Once for each pair of conditions selected from the r conditions it satisfies.)

4) $\binom{r}{3}$ times in $\sum_{1 \le i < j \le k \le t} N(c_i c_j c_k)$. (why?)
…………………………

(r + 1) $\binom{r}{r}$ = 1 time in $\sum N(c_1 c_2 \ldots c_{i_r})$, where the summation is taken over all selections of size r from the t conditions.

Consequently, on the right-hand side of Eq. (2), x is counted

By the binomial theorem. Therefore, the two sides of Eq. (2) count the same elements from S, and the equality is verified.

An immediate corollary of this principle is given as follows:

Corollary 8.1

Under the hypotheses of Theorem 8.1, the number of elements in S That satisfy at least one of the conditions c_i, where $1 \le i \le t$, is given by N (c_1 or c_2 or … or c_t) = N – $\overline{N}$.

Before solving some examples, we examine some further notation for simplifying the statement of Theorem 8.1.

We write

$$S_0 = N$$
$$S_1 = [N(c_1) + N(c_2) + \ldots + N(c_t)],$$
$$S_2 = [N(c_1c_2) + N(c_1c_3) + \ldots + N(c_1c_t) + N(c_2c_3) + \ldots + N(c_{t-1}c_t)],$$

and, in general, $S_k = \sum N(c_{i_1} c_{i_2} \ldots c_{i_k}), 1 \le k \le t,$

Where the summation is taken over all selections of size k from the collection of t conditions.

Hence S_k has $\binom{t}{k}$ summands in it.

Using this notation we can rewrite the result in Eq. (2) as

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + \ldots + (-1)^t S_t.$$

Now let us look at how this principle is used to solve certain enumeration problems.

Example 8.4

Determine the number of positive integers n where $1 \le n \le 100$ and N is not divisible by 2, 3, or 5.

Here S = {1, 2, 3, . . . , 100} and N = 100. For n Є S, n satisfies

a) condition c1 if n is divisible by 2.
b) condition c2 if n is divisible by 3, and
c) condition c3 if n is divisible by 5.

Then the answer to this problem is $\overline{N} = (\overline{c}_1\overline{c}_2\overline{c}_3)$.

As in Section 5.2 we use the notation $\lfloor r \rfloor$ to denote the greatest integer less than or equal to r, for any real number r. this function proves to be helpful in this problem as we find that

$N(c_1) = \lfloor 100/2 \rfloor = 50$ [since the 50 (= $\lfloor 100/2 \rfloor$)] positive integers 2, 4, 6, 8, . . . , 96, 98 (= 2 . 49), 100 (= 2 . 50) are divisible by 2]; $N(c_2) = \lfloor 100/3 \rfloor = \lfloor 33\ \ 1/3 \rfloor = 33$ [since the 33 (=) positive integers 3, 6, 9, 12, . . . , 96 (=3 . 32), 99(= 3 . 33) are divisible by 3]; $N(c_3) = \lfloor 100/5 \rfloor = 20$; N(c1c2) = $\lfloor 100/6 \rfloor$ = 16 [since there are 16 (= $\lfloor 100/6 \rfloor$) elements in S that are divisible by both 2 and 3 – hence divisible by lcm (2, 3) = 2 . 3 = 6];

N(c1c3) = $\lfloor 100/10 \rfloor$ = 10;

N(c2c3) = $\lfloor 100/15 \rfloor$ = 6; and

N(c1c2c3) = $\lfloor 100/30 \rfloor$ =3.

Applying the Principle of Inclusion and Exclusion, we find that

$$N(\overline{c}_1\overline{c}_2\overline{c}_3) = S_0 - S_1 + S_2 - S_3 = N - [N(c_1) + N(c_2) + N(c_3)]$$
$$+ [N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] - N(c_1c_2c_3)$$
$$= 100 - [50 + 33 + 20] + [16 + 10 + 6] - 3 = 26.$$

(These 26 numbers are 1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 77, 79, 83, 89, 91, and 97.)

Example 8.5

In chapter 1 we found the number of nonnegative integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 18$. We now answer the same question with the extra restriction that $x_i \leq 7$, for all $1 \leq i \leq 4$.

Here S is the set of solutions of $x_1 + x_2 + x_3 + x_4 = 18$, with $0 \leq x_i$ for all $1 \leq i \leq 4$.
So $|S| = N = S_0 = \binom{4+18-1}{18} = \binom{21}{18}$

We say that a solution x_1, x_2, x_3, x_4 satisfies condition c_i, where $1 \leq i \leq 4$, if $x_i > 7$ (or $x_i \geq 8$). The answer to the problem is then $\overline{N} = (\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4})$

Here by symmetry $N(c_1) = N(c_2) = N(c_3) = N(c_4)$. To compute $N(c_1)$, We consider the integer solutions for $x_1 + x_2 + x_3 + x_4 = 10$, with each $x_i \geq 0$ for all $1 \leq i \leq 4$, then we add 8 to the value of x_1 and get the solutions of $x_1 + x_2 + x_3 + x_4 = 18$ that satisfy condition c_1.

Hence $N(c_i) = \binom{4+10-1}{10} = \binom{5}{2}, \text{and } S_1 = \binom{4}{1}\binom{13}{10}$

Likewise, $N(c_1c_2)$ is the number of integer solutions of $x_1 + x_2 + x_3 + x_4 = 2$, where $x_i \geq 0$ for all $1 \leq i \leq 4$. So

$N(c_1c_2) = \binom{4+2-1}{2} = \binom{5}{2}$, and $S_2 = \binom{4}{2}\binom{5}{2}$.

Since $N(c_ic_jc_k) = 0$ for every selection of three conditions, and $N(c_1c_2c_3c_4) = 0$, we have

$$N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\,\overline{c_4}) = S_0 - S_1 + S_2 - S_3 + S_4 = \binom{21}{18} - \binom{4}{1}\binom{13}{10} + \binom{4}{2}\binom{5}{2} - 0 + 0 = 246$$

So of the 1330 nonnegative integer solutions of $x_1 + x_2 + x_3 + x_4 = 18$, Only 246 of them satisfy $x_i \leq 7$ for each $1 \leq i \leq 4$.

Our next example establishes the formula conjectured in Section 5.3 For counting onto functions.

Example 8.6

For finite sets A, B where $|A| = m \geq n = |B|$, let $A = \{a_1, a_2, \ldots, a_m\}$, $B = \{b_1, b_2, \ldots, b_n\}$, and S = the set of all functions $f: A \rightarrow B$. Then $N = S0 = |S| = n^m$.

For all $1 \leq i \leq n$, let c_i denote the condition on S where a function f: A → B satisfies c_i if b_i not in the range of f. (Note the difference between c_i here and c_i in Examples 8.4 and 8.5.) Then $N(\overline{c}_i)$ is the number of functions in S that have b_i in their range, and $N(\overline{c}_1 \overline{c}_2 \ldots \overline{c}_n)$ counts the number of onto functions f: A → B.

For all $1 \leq i \leq n$, $N(\overline{c}_i) = (n - 1)^m$, because each of element of B, Except b_i, can be used as the second component of an ordered pair for a function f: A → B, whose range does not include bi. Likewise, for all $1 \leq i < j \leq n$, there are $(n - 2)^m$ functions f: A → B, whose range contains neither bi or bj. From these observations we have $S1 = [N(c_1) + N(c_2) + \ldots + N(c_n)] = n\,(n - 1)^m =$ (n - 1)m, and $S_2 = [N(c_1c_2) + N(c_1c_3) + \ldots + N(c_1c_n) + N(c_2c_3) + \ldots + N(c_2c_n) + \ldots + N(c_{n-1}c_n)] = \binom{n}{2}$(n – 2)m. In general, for each $1 \leq k \leq n$,

$$S_k = \sum_{1 \leq i_1 \leq i_2 < \ldots < i_k \leq n} N(c_{i_1} c_{i_2} \ldots c_{i_k}) = \binom{n}{k}(n-k)^m .$$

It then follows by the Principle of Inclusion and Exclusion that the number of onto functions A and B is

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\ldots\overline{c}_n) = S_0 - S_1 + S_2 - S_3 + \ldots + (-1)^n S_n$$

$$= n^m - \binom{n}{1}(n-1)^m + \binom{n}{2}(n-2)^m - \binom{n}{3}(n-3)^m$$

$$+ \ldots + (-1)^n (n-n)^m = \sum_{i=0}^{n}(-1)^i \binom{n}{i}(n-i)^m$$

$$= \sum_{i=0}^{n}(-1)^i \binom{n}{n-i}(n-i)^m .$$

Before we finish discussing this example, let us note that

$$\sum_{i=0}^{n}(-1)^i \binom{n}{n-i}(n-i)^m$$

Can also be evaluated even if $m < n$ further more, for $m < n$, the expression

$$N(\overline{c}_1\overline{c}_2\overline{c}_3\ldots\overline{c}_n)$$

Still counts the number of functions f: A → B, where $|A| = m$, $|B| = n$ And each element of B is in the range of f. But now this number is 0.

For example, suppose that m = 3 < 7 = n. Then $N(\bar{c}_1\bar{c}_2\bar{c}_3...\bar{c}_n)$ counts the number of onto functions f: A → B, for |A| = 3, |B| = 7. We know this number is 0, and we also find that

$$\sum_{i=0}^{7}(-1)^i\binom{7}{7-i}(7-i)^3 = \binom{7}{7}7^3 - \binom{7}{6}6^3 + \binom{7}{5}5^3 - \binom{7}{4}4^3 + \binom{7}{3}3^3 - \binom{7}{2}2^3 + \binom{7}{1}1^3 - \binom{7}{0}7^3$$

Hence, for all m, n Є Z+, if m < n, then $\sum_{i=0}^{n}(-1)^i\binom{n}{n-i}(n-i)^m = 0$

We now solve a problem similar to those in chapter 3 that dealt with Venn diagrams.

Example 8.7

In how many ways can the 26 letters of the alphabet be permuted so that none of the patterns car, dog, pun, or byte occurs?

Let S denote the set of all permutations of the 26 letters. Then |S| = 26! For each 1 ≤ i ≤ 4, a permutation in S is said to satisfy condition c_i if the permutation contains the pattern car, dog, pun, or byte, respectively.

In order to compute $N(c_1)$, for example, we count the number of ways the 24 symbols car, b, d, e, f, . . . , p, q, s, t, . . . , x, y, z can be permuted. So N(c1) = 24!, and in a similar way we obtain

$$N(c_2) = N(c_3) = 24!, \qquad while\, N(c_4) = 23!$$

For $N(c_1c_2)$ we deal with the 22 symbols car, dog, b, e, f, h, i, . . . , m, n, p, q, s, t, . . . , x, y, z, which can be permuted in 22! ways.
Hence $N(c_1c_2)$ = 22!, and comparable calculations give

$$N(c_1c_3) = N(c_2c_3) = 22!, \qquad N(c_ic_4) = 21!, \quad i \neq 4.$$

Furthermore, $N(c_1c_2c_3) = 20!, \qquad N(c_ic_jc_4) = 19!, \quad i \leq i < j \leq 3,$

$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = 17!$$

So the number of permutations in S that contain none of the given patterns is

$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4) = 26! - [3(24!) + 23!] + [3(22!) + 3(21!)] - [20! + 3(19!)] + 17!$$

Our next example deals with a number theory problem.

Example 8.8

For n Z^+, n ≥ 2, let ø (n) be the number of positive integers m, where 1 ≤ m < n and gcd (m, n) = 1 – that is, m, n are relatively prime. This function is known as Euler's phi function, and it arises in abstract algebra involving enumeration. We find that ø(2) = 1, ø(3) = 2, ø(4) = 3, ø(5) = 4, and ø(6) = 2. for each prime p, ø (p) = p – 1. We would like to derive a formula for ø (n) that is related to n so that we need not make a case-by-case comparison for each m, 1 ≤ m < n, against the integer n,

The derivation of four formula will use the Principle of Inclusion and Exclusion as in Example 8.4. We proceed as follows: For n≥2, use the Fundamental Theorem of Arithmetic to write $n = p_1^{e_1} p_2^{e_2} ... p_t^{e_t}$

Where p_1, p_2, . . . , p_t are distinct primes and $e_i \geq 1$, for all 1 ≤ i < t. We consider the case where t = 4. this will be enough to demonstrate the general idea.

With S = {1, 2, 3, . . . , n}, we have N = S_0 = |S| = n, and for each 1 ≤ i < 4 we say that k Є S satisfies condition c_i if k is divisible by p_i. For 1 ≤ k < n, gcd (k, n) = 1 if k is not divisible by any of the primes p_i, where 1 ≤ i < 4. Hence ø(n) =

For each 1 ≤ i < 4, we have $N(c_i) = n/p_i$; $N(c_ic_j) = n/(p_ip_j)$, for all 1 ≤ i < j ≤ 4. Also, $N(c_ic_jc_l) = n/(p_i\, p_j\, p_l)$, for all 1 ≤ i < j < l ≤ 4, and $N(c_1c_2c_3c_4) = n/(p_1\, p_2\, p_3\, p_4)$. So

$$\phi(n) = S_0 - S_1 + S_2 - S_3 + S_4$$

$$= n - \left[\frac{n}{p_1} + ... + \frac{n}{p_4}\right] + \left[\frac{n}{p_1p_2} + \frac{n}{p_1p_3} + ... + \frac{n}{p_3p_4}\right]$$

$$- \left[\frac{n}{p_1p_2p_3} + ... + \frac{n}{p_2p_3p_4}\right] + \frac{n}{p_1p_2p_3p_4}$$

$$= n\left[1 - \left(\frac{1}{p_1} + ... + \frac{1}{p_4}\right) + \left(\frac{1}{p_1p_2} + \frac{1}{p_1p_3} + ... + \frac{1}{p_3p_4}\right)\right.$$

$$\left. - \left(\frac{1}{p_1p_2p_3} + ... + \frac{1}{p_2p_3p_4}\right) + \frac{1}{p_1p_2p_3p_4}\right]$$

$$= \frac{n}{p_1 p_2 p_3 p_4} [p_1 p_2 p_3 p_4 - (p_2 p_3 p_4 + p_1 p_3 p_4 + p_1 p_2 p_4 + p_1 p_2 p_3)$$

$$+ (p_3 p_4 + p_2 p_4 + p_2 p_3 + p_1 p_4 + p_1 p_3 + p_1 p_2) - (p_4 + p_3 + p_2 + p_1) + 1]$$

$$= \frac{n}{p_1 p_2 p_3 p_4} [(p_1 - 1)(p_2 - 1)(p_3 - 1)(p_4 - 1)]$$

$$= n\left[\frac{p_1 - 1}{p_1} . \frac{p_2 - 1}{p_2} . \frac{p_3 - 1}{p_3} . \frac{p_4 - 1}{p_4}\right] = n \prod_{i=1}^{4} \left(1 - \frac{1}{p_i}\right).$$

In general, $ø(n) = n \Pi_{p/n} (1 - (1/p))$, where the product is taken over all primes p dividing n, when n = p, a prime, ø(n) = ø(p) = p[1–(1/p)] = p – 1, as we observed earlier. If n = 23,100, for example, we find that

ø(23,100) = ø (22 . 3 . 52 . 7 . 11)

= (23,100) (1–(1/2)) (1–(1/3)) (1–(1/5)) (1–(1/7)) (1–(1/11))

= 4800.

The Euler phi function has many interesting properties. We shall investigate some of them in the exercises for this section and in the Supplementary Exercises.

The next example provides another encounter with the circular arrangements introduced in Chapter 1.

Example 8.9

Six married couples are to be seated at circular table. In how many ways can they arrange themselves so that no wife sits next to her husband? (Here, as in Example 1.24, two seating arrangements are considered the same if one is rotation of the other.)

For $1 \leq i \leq 6$, we let c_i, for instance, we consider arranging 11 distinct objects – namely, couple 1 (considered as one object) and other 10 people. Eleven distinct objects can be arranged around a circular table in (11–1)! = 10! ways. However, here $N(c_1) = 2(10!)$, where the 2 takes into account whether the wife in couple 1 is seated to the left or right of her husband. Similarly, $N(c_i) = 2(10!)$, for $2 \leq i \leq 6$, and $S_1 = \binom{6}{1} 2(10!)$.

Continuing, let us now compute $N(c_ic_j)$, for $1 \le i < j \le 6$. Here we are arranging 10 distinct objects – couple i (considered as on object), couple j (likewise considered as one object), and the other eight people. Ten distinct objects can be arranged around a circular table in $(10 - 1)! = 9!$ Ways. So here $N(c_ic_j) = 22(9!)$ because there are two ways for the wife in couple i to be seated next to her husband, and two ways for the wife in couple j to be seated next to her husband.

Consequently, $S_2 = \binom{6}{2} 2^2(9!)$.

$N(c_1c_2c_3) = 2^3(8!)$, $S_3 = \binom{6}{3}2^3(8!)$ $\qquad$ $N(c_1c_2c_3c_4) = 2^4(7!)$, $S_4 = \binom{6}{4}2^4(7!)$

$N(c_1c_2c_3c_4c_5) = 2^5(6!)$, $S_5 = \binom{6}{5}2^5(6!)$ $\qquad$ $N(c_1c_2c_3c_4c_5c_6) = 2^6(5!)$, $S_6 = \binom{6}{6} 2^6(5!)$.

With S_0 (the total number of arrangements of the 12 people) = (12–1)! = 11!, we find that the number of arrangements where no couple is seated side by side is

$$N\left(\bar{c}_1\bar{c}_2\ldots\bar{c}_6\right) = \sum_{i=10}^{6}(-1)^i S_i = \sum_{i=10}^{6}(-1)^i\binom{6}{i}2^i(11-i)!$$

= 39,916,800 – 43,545,600 + 21,772,800 – 6,451,200 + 1,209,600 – 138,240 + 7680
= 12,771,840.

Our final example recalls some of graph theory we studied in Chapter 7.

Example 8.10

In certain area of the countryside are five villages. An engineer is to devise a system of two-way roads so that after the system is completed, no village will be isolated. In how many ways can he do this?

Calling the villages a, b, c, d, and e, we seek the number of loop-free undirected graphs on these vertices, where no vertex is isolated. Consequently, we want to count situations such as those illustrated in parts (a) and (b) of Fig. 8.3, but not situations such as those shown in parts (c) and (d).

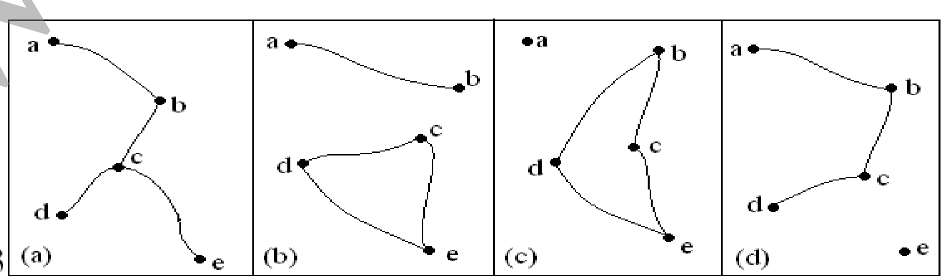


Figure 8.3

Let S be the set of loop-free undirected graphs G on V = {a, b, c, d, e}. Then N = S_0 = |S| =2^{10} because there are $\binom{5}{2}$ = 10 possible two-way roads for these five villages, and each road can be either included or excluded.

For each $1 \le i \le 5$, let ci be the condition that a system of these roads isolates village a, b, c, d, and e, respectively. Then the answer to the problem is $N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4\bar{c}_5)$

For condition c_1 village a is isolated, so we consider the six edges (roads) {b, c}, {b, d}, {b, e}, {c, d}, {c, e}, {d, e}. With two choices for each edge–namely, put the edge in the graph or leave the edge out–we find that $N(c_1) = 2^6$. Then by symmetry $N(c_i) = 2^6$ for all $2 \le i \le 5$, so $S_1 = \binom{5}{1}\ 2^6$.

When villages a and b are to be isolated, each of the edges {c, d}, {d, e}, {c, e} may be put in or left out of our graph. This results in 2^3 possibilities, so $N(c_1c_2) = 23$, and $S_2 = \binom{5}{2}2_3$.

Similar arguments tell us that $N(c_1c_2c_3) = 2^1$ and $S_3 = \binom{5}{3}\ 2^1$; $N(c_1c_2c_3c_4) = 2^0$ and $S_4 = \binom{5}{4}\ 2^0$; $N(c_1c_2c_3c_4c_5) = 2^0$ and $S_5 = \binom{5}{5}2^0$. Consequently,

$$N(\bar{c}_1\bar{c}_2\bar{c}_3\bar{c}_4\bar{c}_5) = 2^{10} - \binom{5}{1}2^6 + \binom{5}{2}2^3 - \binom{5}{3}2^1 + \binom{5}{4}2^0 - \binom{5}{5}2^0 = 768$$

Example 8.11

Find the number of integers between 1 and 10,000 inclusive, which are divisible by none of 5, 6 or 8.

Let P_1 be the property that an integer is divisible by 5, P_2 the property that an integer is divisible by 6, P_3 the property that an integer is divisible by 8. Let A be the set consisting of the first 10,000 integers. Let A_i be the set consisting of those integers in A with property P_i, for i = 1, 2, 3. The problem is to find the number of integers in $\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3$. Now $|A_1| = \lfloor 10,000/5 \rfloor = 2000$, $|A_2| = \lfloor 10,000/6 \rfloor = 1666$, $|A_3| = \lfloor 10,000/8 \rfloor = 1250$. Integers in the set $A_1 \cap A_2$ are divisible by both 5 and 6. Note that an integer is divisible by both 5 and 6 if it is divisible by their lcm {5, 6} = 30.

Also lcm {5, 8} = 40, lcm {6, 8} = 24. then $|A1 \cap A2| = \lfloor 10,000/30 \rfloor = 333$.$|A1 \cap A3| = \lfloor 10,000/40 \rfloor = 250$, $|A2 \cap A3| = \lfloor 10,000/24 \rfloor = 416$.

Also $|A1 \cap A2 \cap A3| = \lfloor 10,000/120 \rfloor = 83$, since lcm {5, 6, 8} = 120.

Now by Principle of Inclusion and Exclusion, the number of integers between 1 and 10,000 that are divisible by none of 5, 6 and 8 equals.

$$|A1 \cap A2 \cap A3| = |A| - (|A1| + |A2| + |A3|) + (|A1 \cap A2| + |A2 \cap A3| + |A3 \cap A1|) - |A1 \cap A2 \cap A3|$$

$$= 10{,}000 - (2000 + 1666 + 1250) + (333 + 250 + 416) - 83$$

$$= 6000.$$

Example 8.12

Determine the number of permutations of the letters J, N, U, I, S, G, R, E, A, T such that none of the words JNU IS and GREAT occur as consecutive letters (that is, permutations such as JNTUISGREAT, ISJNUGREAT, UNJGREATSI etc are not allowed).

Let A be the set of all permutations of the 10 letters given. Let P_1 be the property that a permutation in A contains the word JNU as Consecutive letters, let P_2 be the property that a permutation contains the word IS and let P_3 be property that a permutation contains the GREAT. Let A_i be the set of those permutations in A satisfying the property P_i for i = 1, 2, 3. The problem is to find the number of permutations in Now $|A| = 10! =$ 3,628,800. The set A_1 contains the permutations of the 8 symbols JNU, I, S, G, R, E, A, T so $|A_1| = 8!$. Similarly A_2 contains permutations of the 9 symbols J, N, U , IS, G, R, E, A, T so $|A_2| = 9! = 362{,}880$. Similarly A_3 contains permutations of the 6 symbols J, N, U , I, S, GREAT so $|A_3| = 6! = 720$. Also $|A_1 \cap A_2| = 7! = 5040$, since $A_1 \cap A_2$ contains permutations of the 7 symbols JNU, IS, G, R, E, A, T. Also $|A_1 \cap A_3| = 4! = 24$, since $A_1 \cap A_3$ contains permutations of the 4 symbols JNU, I, S, GREAT. Finally $|A_1 \cap A_2 \cap A_3| = 3! = 6$, since $A_1 \cap A_2 \cap A_3$ contains the permutations of the three symbols JNU, IS, GREAT. Using the Principle of Inclusion – Exclusion we have

$$\begin{aligned} |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| &= 3{,}628{,}800 - (40{,}320 + 362{,}880 + 720) \\ &\quad + (5040 + 24 + 120) - 6 \\ &= 3{,}230{,}058 \end{aligned}$$

Example 8.12

During a 12-week conference of mathematics, the vice- chancellor (V.C.) met his seven friends from college. During the conference, V.C. met each friend at lunch 35 times, every pair of them 16 times, Every trio eight times, every foursome four times, each set of five twice, and each set of six once, but never all seven at once. If he had lunch every day during the 84 days of conference, did he ever have lunch alone.

For $1 \le i \le 7$, let ci denote the situation where the i^{th} friend had lunch with V.C. Then N = 12 weeks X 7 days = 84 days = S_0. since V.C. met each friend at lunch 35 times, $N(c_i) = 35$ for any i. Since there are $\binom{7}{1}$ ways of one friend out of seven, we have $S_1 = \binom{7}{1} 32$.

Also since V.C. met every pair 16 times, we get $S_2 = \binom{7}{2} 16$. Similarly $S_3 = \binom{7}{3} 8$, $S_4 = \binom{7}{4} 4$, $S_5 = \binom{7}{5} 2$, $S_6 = \binom{7}{6} 1$, $S_7 = \binom{7}{7} 0$. Thus by Principle of Inclusion – Exclusion, V.C. not having lunch with any of the seven friends on any of the 84 days

$$= S0 - S1 + S2 - S3 + S4 - S5 + S6 - S7$$

$$= 84 - 7(35) + 21(16) - 35(8) + 35(4) - 21(2) + 7.1 - 1.0$$

$$= 0$$

8.2 Generalizations of the Principle

Consider a set S with |S| = N, and conditions $c_1, c_2, \ldots, c_t$ satisfied by some of the elements of S. In section 8.1 we saw how the Principle of Inclusion and Exclusion provides a way to determine $N(\overline{c_1}\,\overline{c_2}\ldots\overline{c_t})$, the number of elements in S that satisfy none of the t conditions. If $m \in Z^+$ and $1 \le m \le t$, we now want to determine E_m, which denotes the number of elements in S that satisfy exactly m of the t conditions. (At present we can obtain E_0.)

We can write formulas such as

$$E_1 = N(c_1\overline{c_2}\,\overline{c_3}\ldots\overline{c_t}) + N(\overline{c_1}c_2\overline{c_3}\ldots\overline{c_t}) + \ldots + N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\ldots\overline{c_{t-1}}c_t),$$

and

$$E_2 = N(c_1c_2\overline{c_3}\ldots\overline{c_t}) + N(c_1\overline{c_2}c_3\ldots\overline{c_t}) + \ldots + N(\overline{c_1}\,\overline{c_2}\,\overline{c_3}\ldots\overline{c_{t-2}}c_{t-1}c_t),$$

and although these results do not assist us as much as we should like, they will be a useful starting place as we examine the Venn diagrams for the cases where t = 3 and 4.

For Fig. 8.4, where t = 3, we place a numbered condition and also number each of the individual regions shown. Then E_1 equals the number of elements in regions 2, 3, and 4.

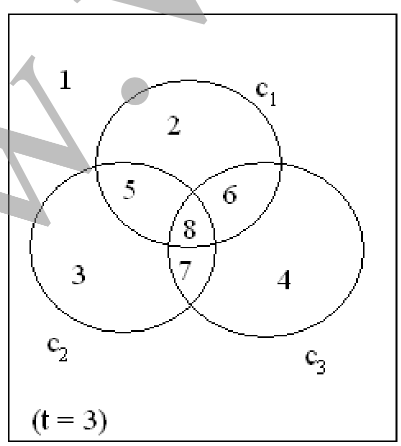


Figure 8.4

But we can also write

$E_1 = N(c_1) + N(c_2) + N(c_3) - 2[N(c_1c_2) + N(c_1c_3) + N(c_2c_3)] + 3N(c_1c_2c_3)$.

In $N(c_1) + N(c_2) + N(c_3)$ we count the elements in regions 5, 6, and 7 twice and those in region 8 three times. In the next term, the elements in regions 5, 6, and 7 are deleted twice. We remove the elements in region 8 six times in $2[N(c_1c_2) + N(c_1c_3) + N(c_2c_3)]$, so we then add on the term $3N(c_1c_2c_3)$ and end up not counting the elements in region 8 at all.

Hence we have $E_1 = S_1 - 2S_2 + 3S_3 = S_1 - \binom{2}{1}S_2 + \binom{3}{2}S_3$.

When we turn to E1, our earlier formula indicates that we want to count the elements of S in regions 5, 6, and 7. From the Venn diagram,

$$E_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) - 3N(c_1c_2c_3)$$
$$= S_2 - 3S_3 = S_2 - \quad S_3,$$

and

$$E_3 = N(c_1c_2c_3) = S_3.$$

In Fig. 8.5, the conditions c_1, c_2, c_3 are associated with circular subsets of S, where as c_4 is paired with the rather irregularly shaped area made up of regions 4, 8, 9, 11, 12, 13, 14, and 16. For each $1 \leq i \leq 4$, Ei is determined as follows:

E_1 [regions 2, 3, 4, 5]:

$$E_1 = [N(c_1) + N(c_2) + N(c_3) + N(c_4)] - 2[N(c_1c_2) + N(c_1c_3)$$
$$+ N(c_1c_4) + N(c_2c_3) + N(c_2c_4) + N(c_1c_4)]$$
$$+ 3[N(c_1c_2c_3) + N(c_1c_2c_4) + N(c_1c_3c_4) + N(c_2c_3c_4)]$$
$$- 4N(c_1c_2c_3c_4)$$
$$= S_1 - 2S_2 + 3S_3 - 4S_4 = S_1 - \quad S_2 + \quad S_3 - \quad S_4.$$

Note: Taking an elements in region 3, we find that it is counted once in E_1 and once in S_1 [in $N(c_3)$]. Taking an elements in region 6, we find that it is not counted in E_1; it is counted twice in S_1 [in both $N(c_2)$ and $N(c_3)$] but removed twice in 2S2 [for it is counted once in S_2 in $N(c_2c_3)$], so overall it is not counted. The reader should now consider an element from region 12 and none from region 16 and show that each contributes a count of 0 to both sides of the formula for E_1.

8.3 Derangements: Nothing Is in Its Place

In elementary calculus the Maclaurin series for the exponential function is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

$$e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = 1 - 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

To five places, $e^{-1} = 0.36788$ and 1 – 1+(1/2!)–(1/3!)+. . . –(1/7!) = 0.36786. Consequently, for all k Є Z+, if k ≥ 7, then $\sum_{n=0}^{k}((-1)^n)/n!$ is very good approximation to e^{-1}.

We find these ideas helpful in working some of the following Examples.

8.16 Example

While at the racetrack, Ralph bets on the ten horses in a race to come in according to how many they are favored. In how many ways can they reach the finish line so that he loses all of his bets?

Removing the words horses and racetrack from the problem, we really want to know in how many ways we can arrange the numbers 1, 2, 3, . . ., 10. So that 1 is not in first place (its natural position), 2 is not in second place (its natural position), . . . , and 10 is not in tenth place (its natural position). These arrangements are called the derangements of 1, 2, 3, . . . , 10.

The Principle of Inclusion and Exclusion provides the key to calculating the number of derangements. For each 1 ≤ i ≤ 10, an arrangement of 1, 2, 3, . . . , 10 is said to satisfy condition c_i if integer i is in the i^{th} place. We obtain the number of derangements, denoted by d_{10}, as follows:

$$d_{10} = N(\bar{c}_1\bar{c}_2\bar{c}_3\ldots\bar{c}_{10}) = 10! - \binom{10}{1}9! + \binom{10}{2}8! - \binom{10}{3}7! + \ldots + \binom{10}{10}0!$$

$$= 10!\left[1 - \binom{10}{1}(9!/10!) + \binom{10}{2}(8!/10!) - \binom{10}{3}(7!/10!) + \ldots + \binom{10}{10}(0!/10!)\right]$$

$$= 10![1 - 1 + (1/2!) - (1/3!) + \ldots + (1/10!)] = (10!)(e^{-1}).$$

The sample space here consists of the 10! Ways the horses can finish. So the probability that Ralph will lose every bet is approximately $(10!)(e^{-1})/(10!) = e^{-1}$. This probability remains (more or less) the same if the number of horses in the race is 11, 12, ... On the other hand, foe n horses, where $n \geq 10$, the probability that our gambler wins at least one of his bets is approximately $1 - e^{-1} = 0.63212$

8.18 Example

Peggy has seven books to review for the C–H Company, so she hires seven people to review them. She wants two reviews per book, so the first week she gives each person one book to read and then redistributes the books at the start of the second week. In how many ways can she make these two distributions so that she gets two reviews (by different people) of each book?

She can distribute the books in 7! ways the first week. Numbering both the books and the reviews (for the first week) as 1, 2, . . ., 7, for the second distribution she must arrange

These numbers so that none of them is in its natural position. This she can do in d7 ways. By the rule of product, she can make the two distributions in $(7!)d7 = (7!)2(e^{-1})$ ways.

The number of derangements of a set with n elements is

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right], \quad for\, n > 1$$

For example

$$D_2 = 2!\left[1 - \frac{1}{1!} + \frac{1}{2!}\right] = 1$$

$$D_3 = 3!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right] = 6\left(1 - 1 + \frac{1}{2} - \frac{1}{6}\right) = 2$$

$$D_4 = 9, \quad D_5 = 44, \quad D_6 = 265, \quad D_7 = 1854.$$

8.19 Example

A machine that inserts letters into envelopes goes haywire and inserts letters randomly into envelopes. What is the probability in a group of 100 letters (a) no letter is put into the correct envelope (b) exactly 1 letter is put into the correct envelope (c) exactly 98 letters are put into the correct envelope (d) exactly 99 letters are put into the correct envelope (e) all letters are put into the correct envelopes?

The probability of derangements of n objects is $D_n/n!$. For example

Probability of derangements

n:	2	3	4	5	6	7
Dn/n!:	0.5	0.333	0.375	0.3667	0.36806	0.36786

a) The probability of no letter being put in the correct envelope is $D_n/100!$ Because the number of favorable cases (the derangements) is D_n and the total number of favorable cases is n! = 100!.

b) When exactly 1 letter being put correctly, the number of Derangements for the remaining 99letters is D99. This can happen in $\binom{100}{1}$ or $\binom{100}{99}$ ways. So the probability is

$$\frac{\binom{100}{1}D_{99}}{100!} = \frac{100D_{99}}{100!}$$

c) When exactly 98 letters are put into the right (correct) envelope, the number of derangements for the remaining 2 is D2 = . This can happen in $\binom{100}{98} = \binom{100}{2}$ ways. The required probability is

$$\frac{\binom{100}{2}D_2}{100!} = \frac{\binom{100}{2}}{100!}$$

d) When exactly 99 letters are put in the correct envelope, it is not possible to misplace the remaining one letter. This is also in the right envelope. Thus the probability is zero (It is an Impossible event)

e) When all letters are put in the correct envelope, this can happen only once out of the total 100! cases. Thus the required probability is 1/100!.

8.20 Example

a) List all the derangements of the numbers 1, 2, 3, 4, 5 where the first three numbers are 1, 2, 3 in order.

b) List all the derangements of the numbers 1, 2, 3, 4, 5, 6 where the first three numbers are 1, 2, 3 in some order.

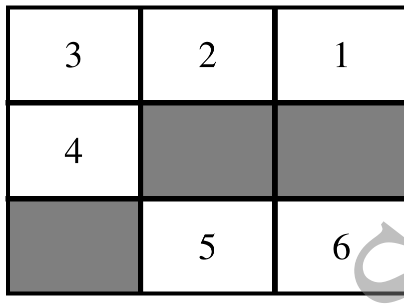
c) When 1, 2, 3 are in some order, there are only two derangements (i) 23,154 and (ii) 31, 254 (other examples include 21,354 and 32,154)

d) There are only for such derangements. For example, one such set is (i) 231,546 (ii) 312,546 (iii) 231,645 (iv) 312,645 (other examples include (i) 213,546 (ii) 321, 546 (iii) 213,645 (iv) 321,645)

8.4 Rook Polynomials

Consider the six-square "chessboard" shown in Fig. 8.6 (Note: The shaded squares are not part of the chessboard.) in chess a piece called a rook or castle is allowed at one turn to be moved horizontally or vertically over as many unoccupied spaces as one wishes. Here a rook in square 3 of the figure could be moved in one turn to squares 1, 2, or 4. A rook at square 5 could be moved to square 6 or square 2 (even though there is no square between squares 5 and 2).

Figure 8.6



For k Є Z^+ we want to determine the number of ways in which k rooks can be placed on the unshaded squares of this chess-board so that no two of them can take each other–that is, no two of them are in the same row or column of the chessboard. This number is denoted by r_k or by r_k (C) if we wish to stress that we are working on particular chessboard C.

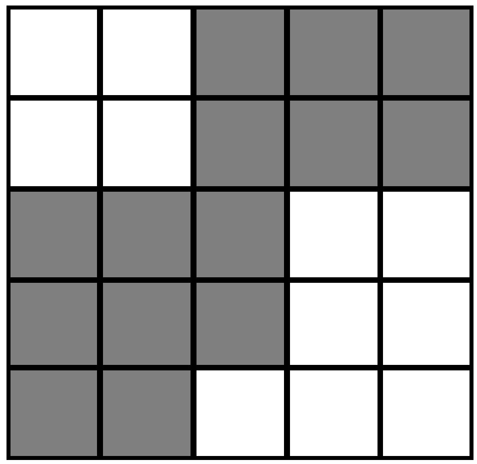
For any chessboard, r1 is the number of squares on the board. Here $r_1 = 6$. Two nontaking rooks can be placed at the following pairs of positions: {1, 4}, {1, 5}, {2, 4}, {2, 6}, {3, 5}, {3, 6}, {4, 5} and {4, 6}, so $r_2 = 8$. Continuing, we find that $r_3 = 2$, using the Locations {1, 4, 5} and {2, 4, 6}; $r_k = 0$, for $k \geq 4$.

With $r_0 = 1$, the rook polynomial, r (C, x), for the chessboard in Fig 8.6 is defined as r (C, x) = $1 + 6x + 8x^2 + 2x^3$. For each $k \geq 0$, the coefficient of x^k is the number of ways we can place k nontaking rooks on chessboard C.

What we have done here (using a case-by-case analysis) soon proves tedious. As the size of the board increases, we have to consider cases wherein numbers such as r_4 and r_5 are nonzero. Consequently, we shall now make some observations that will allow us to make use of small boards and somehow break up a large board into smaller subboards.

The chessboard C in Fig 8.7 is made up of 11 unshaded square. We note that C consists of a 2 X 2 subboard C_1 located in the upper left corner and a seven-square subboard C_2 located in the in the lower right corner. These subboards are disjoint because they have no squares in the same row or column of C.

Figure 8.7



Calculating as we did for our first chessboard, here we find

$r(C_1, x) = 1 + 4x + 2x^2$, $r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$.
$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5 = r(C_1, x) . r(C_2, x)$.

Hence $r(C, x) = r(C_1, x) . r(C_2, x)$. But did this occur by luck or is something happening here that we should examine more closely? For example, to obtain r_3 for C, we send to know in how many ways three nontaking rooks can be placed on board C.

These fall into there cases:

a) All three rooks are on subboard C_2 (and none is on C_1): (2)(1) = 2 ways.

b) Two rooks are on subboard C_2 and one is on C_1: (10)(4) = 40 ways.

c) One rook is on subboard C_2 and two are on C_1: (7)(2) = 14 ways.

Consequently, three nontaking rooks can be placed on board C in (2)(1) + (10)(4) + (7)(2) = 56 ways. Here we see that 56 arises just as the coefficient of x^3 does in the product $r(C_1, x) . r(C_2, x)$.

In general, if C is a chessboard made up of pairwise disjoint subboards $C_1, C_2, \ldots, C_n$, then $r(C, x)\ r(C_2, x) \ldots r(C_n, x)$.

The last result for this section demonstrates the type of principle we have seen in other results in combinatorial and discrete mathematics: Given a large chessboard, break it into smaller subboards whose rook polynomials can be determined by inspection.

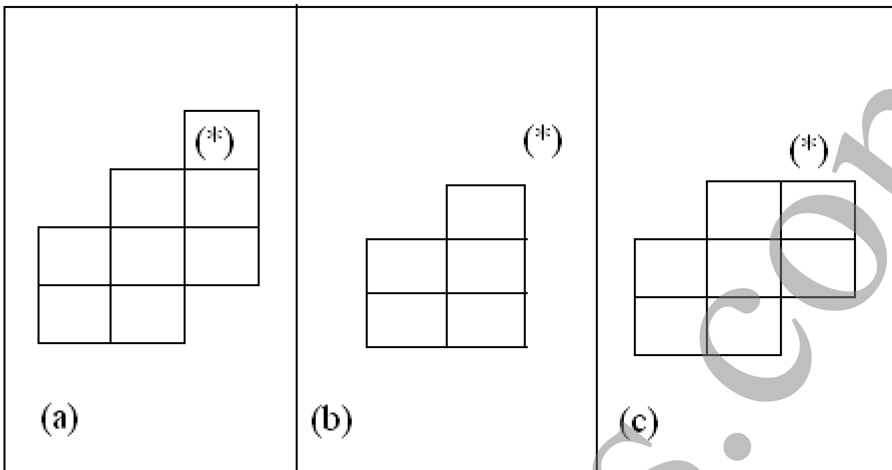


Figure 8.8

Consider chessboard C in Fig. 8.8(a). For $k \geq 1$, suppose we wish to place k nontaking rooks on C. For each square of C, such as the one designated by (*), there are two possibilities to examine.

a) Place one rook on the designated square. Then we remove, as possible locations for the other $k - 1$ rooks, all other squares of C in the same row or column as the designated square. We use C_s to denote the remaining smaller subboard [seen in Fig.8.8 (b)].

b) We do not use the designated square at all. The k rooks are placed on the subboard Ce [C with the one designated square eliminated — as shown in the Fig. 8.8(c)].

Since these two cases are all-inclusive and mutually disjoint,

$$r_k(C) = r_k\text{-}1(C_s) + r_k(C_e).$$

From this we see that

$$r_k(C)x^k = r_{k-1}(C_s)x^k + r_k(C_e)x^k. \qquad (1)$$

If n is the number of squares in the chessboard (here n is 8), Then Eq. (1) is valid for all $1 \leq k \leq n$, and we write

$$\sum_{k=1}^{n} r_k(C)x^k = \sum_{k=1}^{n} r_{k-1}(C_s)x^k + \sum_{k=1}^{n} r_k(C_e)x^k. \qquad (2)$$

For Eq.(2) we realize that the summations may stop before k = n. We have seen cases, as in Fig. 8.6, where r_n and some prior r_k's are 0. The summations start at k = 1, for otherwise we could find ourselves with the term $r_{-1}(C_s)x^0$ in the first summand on the right-hand side of Eq. (2).

$$\sum_{k=1}^{n} r_k(C)x^k = x\sum_{k=1}^{n} r_{k-1}(C_s)x^{k-1} + \sum_{k=1}^{n} r_k(C_e)x^k \qquad (3)$$

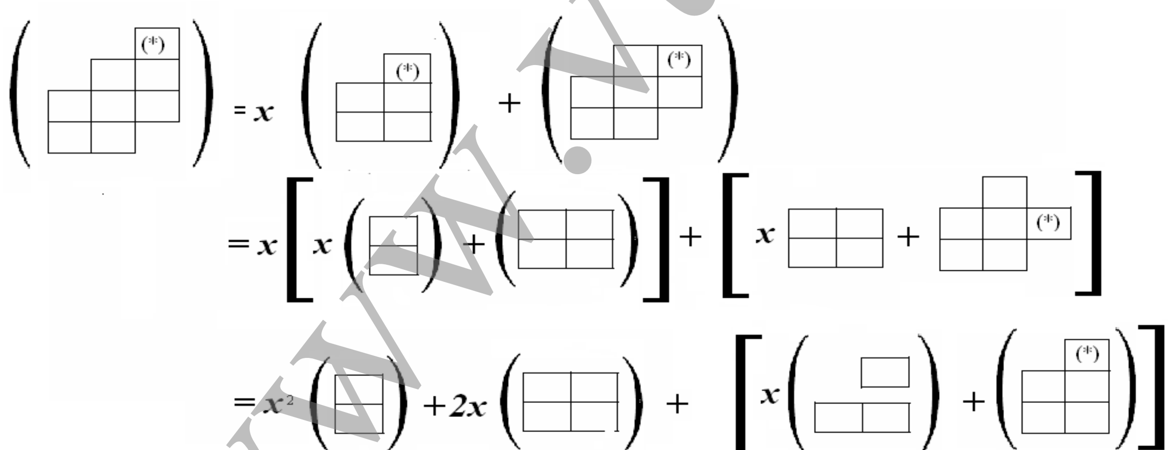
or

$$1+\sum_{k=1}^{n} r_k(C)x^k = x.r(C_s, x) + \sum_{k=1}^{n} r_k(C_e)x^k + 1,$$

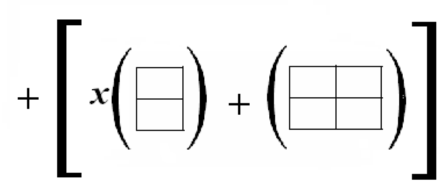
from which it follows that

$$r(C, x) = x.r(C_s, x) + r(C_e, x). \qquad (4)$$

We now use this final equation to determine the rook polynomial for the chessboard shown in part (a) of Fig. 8.8. Each time the idea in Eq. (4) is used, we mark the special square we are using with (*). Parentheses are placed about each chessboard to denote the rook polynomial of the board.



$$= x^2(1+2x) + 2x(1+4x+2x^2) + x(1+3x+x^2)$$



$$= 3x + 12x^2 + 7x^3 + x(1+2x) + (1+4x+2x^2)$$

$$= 1 + 8x + 16x^2 + 7x^3.$$