Learning Objectives

- Learn about recurrence relations
- Learn the relationship between sequences and recurrence relations
- Explore how to solve recurrence relations by iteration
- Learn about linear homogeneous recurrence relations and how to solve them
- Become familiar with linear nonhomogeneous recurrence relations

Sequences and Recurrence Relations

EXAMPLE 8.1.2

Consider the following two sequences:

$$S_1: \ 3, 5, 7, 9, \ldots$$
$$S_2: \ 3, 9, 27, 81, \ldots$$

We can find a formula for the nth term of sequences S_1 and S_2 by observing the pattern of the sequences.

$$S_1: \ 2 \cdot 1 + 1, 2 \cdot 2 + 1, 2 \cdot 3 + 1, 2 \cdot 4 + 1, \ldots$$
$$S_2: \ 3^1, 3^2, 3^3, 3^4, \ldots$$

For S_1, $a_n = 2n + 1$ for $n \geq 1$, and for S_2, $a_n = 3^n$ for $n \geq 1$. This type of formula is called an **explicit formula** for the sequence, because using this formula we can directly find any term of the sequence without using other terms of the sequence. For example, $a_3 = 2 \cdot 3 + 1 = 7$.

EXAMPLE 8.1.3

Let S denote the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

For this sequence, the explicit formula is not obvious. If we observe closely, however, we find that the pattern of the sequence is such that any term after the second term is the sum of the preceding two terms. Now

$$\begin{aligned}
\text{3rd term} &= 2 = 1 + 1 = \text{1st term} + \text{2nd term} \\
\text{4th term} &= 3 = 1 + 2 = \text{2nd term} + \text{3rd term} \\
\text{5th term} &= 5 = 2 + 3 = \text{3rd term} + \text{4th term} \\
\text{6th term} &= 8 = 3 + 5 = \text{4th term} + \text{5th term} \\
\text{7th term} &= 13 = 5 + 8 = \text{5th term} + \text{6th term}
\end{aligned}$$

Hence, the sequence S can be defined by the equation

$$f_n = f_{n-1} + f_{n-2} \qquad (8.1)$$

for all $n \geq 3$ and

$$\begin{aligned} f_1 &= 1, \\ f_2 &= 1. \end{aligned} \qquad (8.2)$$

EXAMPLE 8.1.4

Consider the function $f : \mathbb{N}^0 \to \mathbb{Z}^+$ defined by

$$f(0) = 1,$$
$$f(n) = nf(n-1) \quad \text{for all } n \geq 1.$$

Then

$$f(0) = 1 = 0!,$$
$$f(1) = 1 \cdot f(0) = 1 = 1!,$$
$$f(2) = 2 \cdot f(1) = 2 \cdot 1 = 2 = 2!,$$
$$f(3) = 3 \cdot f(2) = 3 \cdot 2 \cdot 1 = 6 = 3!,$$

and so on. Here $f(n) = nf(n-1)$ for all $n \geq 1$ is the recurrence relation, and $f(0) = 1$ is the initial condition for the function f. Notice that the function f is nothing but the factorial function, i.e., $f(n) = n!$ for all $n \geq 0$.

Sequences and Recurrence Relations

Let us consider the function f as given in (8.3). If we write $a_n = f(n)$, then (8.3) translates into the following equation:

$$a_n = 2a_{n-1} + a_{n-2} \quad \text{for all } n \geq 2.$$

That is, a_n is defined in terms of a_{n-1} and a_{n-2}. As remarked previously, such an equation is called a recurrence relation. Moreover, (8.4) translates into $a_0 = 5$ and $a_1 = 7$. These are called the initial conditions for the recurrence relation.

DEFINITION 8.1.5

A **recurrence relation** for a sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ is an equation that relates a_n to some of the terms $a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}$ for all integers n with $n \geq k$, where k is a nonnegative integer. The **initial conditions** for the recurrence relation are a set of values that explicitly define some of the members of $a_0, a_1, a_2, \ldots, a_{k-1}$.

The equation

$$a_n = 2a_{n-1} + a_{n-2} \quad \text{for all } n \geq 2,$$

as defined above, relates a_n to a_{n-1} and a_{n-2}. Here $k = 2$. So this is a recurrence relation with initial conditions $a_0 = 5$ and $a_1 = 7$.

Example 8.1.9

Number of subsets of a finite set. Let s_n denote the number of subsets of a set A with n elements, $n \geq 0$. In Worked-Out Exercise 9 (Chapter 2, page 144), we proved that

$$s_0 = 1,$$
$$s_n = 2s_{n-1}, \quad \text{if } n > 0$$

Hence, a recurrence relation for the sequence $s_0, s_1, s_2, s_3, s_4, \ldots$ is

$$s_n = 2s_{n-1}, \quad n \geq 1$$

and an initial condition is $s_0 = 1$.

Example 8.1.10

Compound Interest. Sam received a yearly bonus and deposited \$10,000 in a local bank yielding 7% interest compounded annually. Sam wants to know the total amount accumulated after n years. Let A_n denote the total amount accumulated after n years. Let us determine a recurrence relation and initial conditions for the sequence $A_0, A_1, A_2, A_3, \ldots$.

The amount accumulated after one year is the initial amount plus the interest on the initial amount. Now A_{n-1} is the amount accumulated after $n - 1$ years. This implies that the amount at the beginning of nth year is A_{n-1}. It follows that the total amount accumulated after n years is the amount at the beginning of the nth year plus the interest on this amount. Because the interest rate is 7%, the interest earned during the nth year is $(0.07)A_{n-1}$. Hence,

$$A_n = A_{n-1} + (0.07)A_{n-1}$$
$$= 1.07A_{n-1}, \quad n \geq 1,$$
$$A_0 = 10000.$$

- **Tower of Hanoi**
- In the nineteenth century, a game called the Tower of Hanoi became popular in Europe. This game represents work that is under way in the temple of Brahma.
- There are three pegs, with one peg containing 64 golden disks. Each golden disk is slightly smaller than the disk below it.
- The task is to move all 64 disks from the first peg to the third peg.

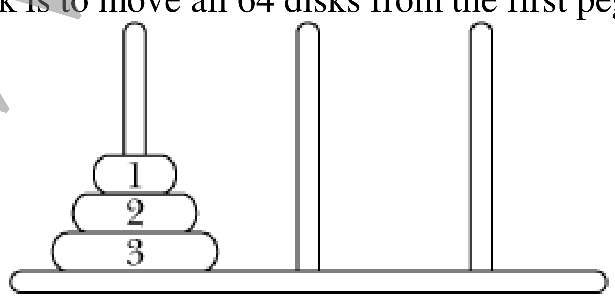

FIGURE 8.1 Tower of Hanoi problem with three disks

- The rules for moving the disks are as follows:
    1. Only one disk can be moved at a time.
    2. The removed disk must be placed on one of the pegs.
    3. A larger disk cannot be placed on top of a smaller disk.

- The objective is to determine the minimum number of moves required to transfer the disks from the first peg to the third peg.

- First consider the case in which the first peg contains only one disk.
    - The disk can be moved directly from peg 1 to peg 3.

- Consider the case in which the first peg contains two disks.
    - First move the first disk from peg 1 to peg 2.
    - Then move the second disk from peg 1 to peg 3.
    - Finally, move the first disk from peg 2 to peg 3.

- Consider the case in which the first peg contains three disks and then generalize this to the case of 64 disks (in fact, to an arbitrary number of disks).
    - Suppose that peg 1 contains three disks. To move disk number 3 to peg 3, the top two disks must first be moved to peg 2. Disk number 3 can then be moved from peg 1 to peg 3. To move the top two disks from peg 2 to peg 3, use the same strategy as before. This time use peg 1 as the intermediate peg.
    - Figure 8.2 shows a solution to the Tower of Hanoi problem with three disks.

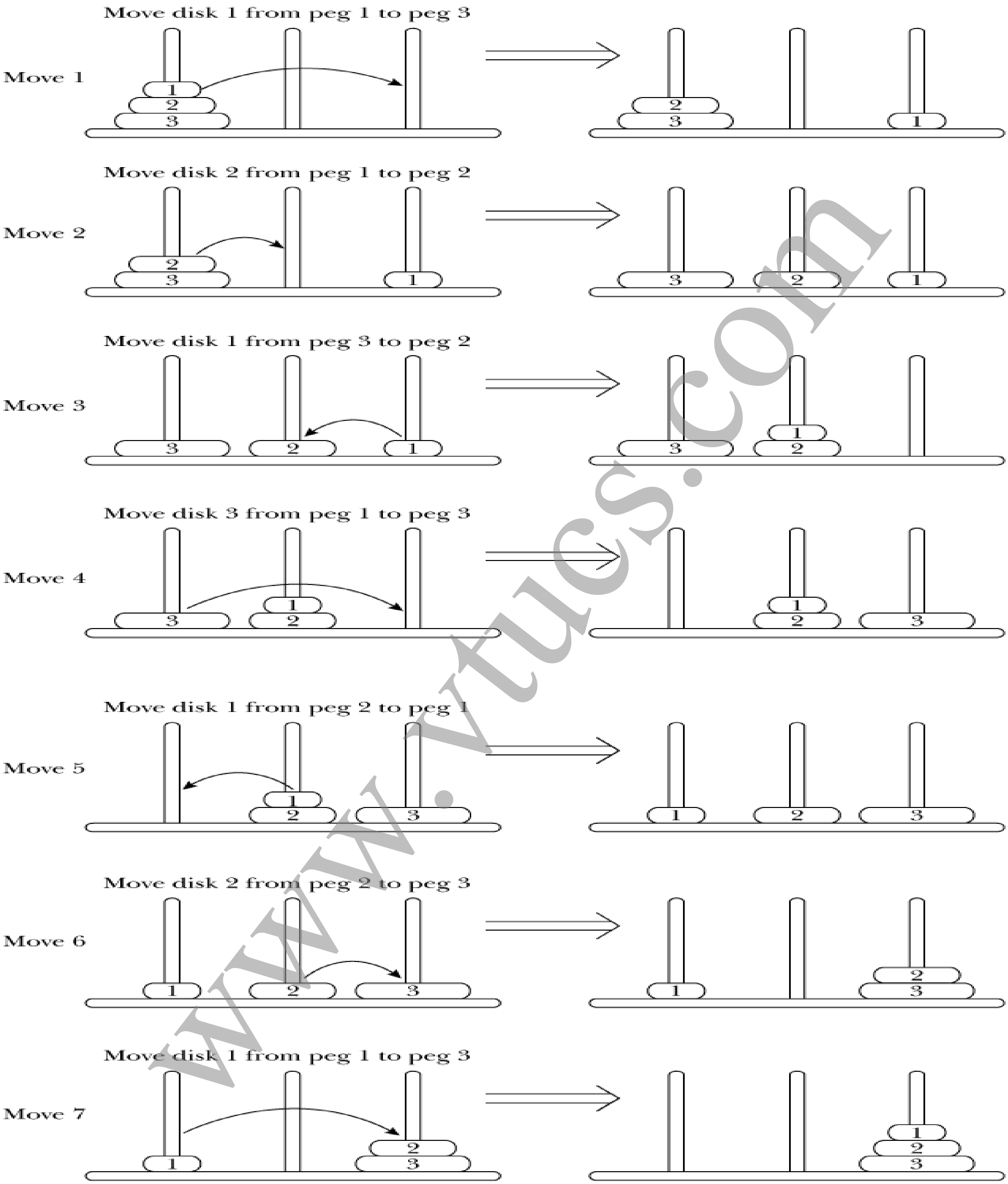

FIGURE 8.2 A solution to the Tower of Hanoi problem with three disks

- Generalize this problem to the case of 64 disks. To begin, the first peg contains all 64 disks. Disk number 64 cannot be moved from peg 1 to peg 3 unless the top 63 disks are on the second peg. So first move the top 63 disks from peg 1 to peg 2, and then move disk number 64 from peg 1 to peg 3. Now the top 63 disks are all on peg 2.

- To move disk number 63 from peg 2 to peg 3, first move the top 62 disks from peg 2 to peg 1, and then move disk number 63 from peg 2 to peg 3. To move the remaining 62 disks, follow a similar procedure.

- In general, let peg 1 contain $n \geq 1$ disks.
  1. Move the top $n - 1$ disks from peg 1 to peg 2 using peg 3 as the intermediate peg.
  2. Move disk number n from peg 1 to peg 3.
  3. Move the top $n - 1$ disks from peg 2 to peg 3 using peg 1

Let c_n denote the number of moves required to move n disks, $n \geq 0$, from peg 1 to peg 3. Step (1) requires us to move the top $n - 1$ disks from peg 1 to peg 2, which requires c_{n-1} moves. Step (2) requires us to move the nth disk from peg 1 to peg 3, which requires 1 move. Step (3) requires us to move $n - 1$ disks from peg 2 to peg 3, which requires c_{n-1} moves. Thus, it follows that

$$c_n = 2c_{n-1} + 1, \quad \text{if } n > 1, \tag{8.5}$$

and

$$c_1 = 1. \tag{8.6}$$

Now (8.5) is a recurrence relation for the sequence $\{c_n\}_{n=1}^{\infty}$ with the initial condition given by (8.6).

DEFINITION 8.1.13

Suppose a recurrence relation for a sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$, is given. By a *solution of the recurrence relation* we mean to obtain an explicit formula for a_n, i.e., to find an expression for a_n that does not involve any other a_i.

Let S be the sequence $\{a_n\}_{n=0}^{\infty}$, where

$$a_n = 7a_{n-1} - 6a_{n-2} \quad \text{for all } n \geq 2. \tag{8.8}$$

Because a_n is defined in terms of the preceding terms a_{n-1} and a_{n-2}, Equation (8.8) is a recurrence relation.

Let us show that $a_n = 5 = 5 + 0 \cdot n$ is a solution of Equation (8.8). Here $a_0 = 5$, $a_1 = 5$, $a_2 = 5, \ldots, a_n = 5$, and so on. Let us evaluate the right side of Equation (8.8), i.e.,

$$7a_{n-1} - 6a_{n-2} = 7 \cdot 5 - 6 \cdot 5 = 35 - 30 = 5 = a_n.$$

Hence, $a_n = 5, n \geq 0$ is a solution of the recurrence relation (8.8).

Now let $a_n = 6^n$. Here $a_0 = 6^0 = 1$, $a_1 = 6^1 = 6$, $a_2 = 6^2 = 36, \ldots, a_{n-2} = 6^{n-2}$, $a_{n-1} = 6^{n-1}$, $a_n = 6^n$, and so on. Let us evaluate the right side of Equation (8.8), using the terms of this sequence. We have

$$\begin{aligned} 7a_{n-1} - 6a_{n-2} &= 7 \cdot 6^{n-1} - 6 \cdot 6^{n-2} \\ &= 7 \cdot 6^{n-1} - 6^{n-1} \\ &= (7 - 1) \cdot 6^{n-1} \\ &= 6 \cdot 6^{n-1} \\ &= 6^n \\ &= a_n. \end{aligned}$$

Therefore, $a_n = 6^n, n \geq 0$ is also a solution of the recurrence relation (8.8).

Note that the expression $a_n = 2^n$, $n \geq 0$ is not a solution of Equation (8.8).

Linear Homogenous Recurrence Relations

DEFINITION 8.2.1

Let $a_0, a_1, a_2, \ldots, a_n, \ldots$ be a sequence of numbers. A **linear homogeneous recurrence relation** of order k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}, \tag{8.31}$$

where $c_k \neq 0$ and $c_1, c_2, c_3, \ldots,$ and c_k are constants.

Linear Homogenous Recurrence Relations

EXAMPLE 8.2.2

Consider the following recurrence relations.

(i) $a_n = 3a_{n-1} + a_{n-2}$
(ii) $a_n = 3a_{n-1} + 5$
(iii) $a_n = 3a_{n-1} + a_{n-2} \cdot a_{n-3}$
(iv) $a_n = 3a_{n-1} + a_{n-2} + \sqrt{2}a_{n-3}$
(v) $a_n = 3a_{n-1} + na_{n-2}$

Recurrence relations (i), (ii), (iii), and (iv) are recurrence relations with constant coefficients. Recurrence relation (v), $a_n = 3a_{n-1} + na_{n-2}$, is not a relation with constant coefficients. Notice that (i) is a linear homogeneous recurrence

Linear Homogenous Recurrence Relations

DEFINITION 8.2.3

A sequence $s_0, s_1, s_2, \ldots, s_n, \ldots$ is said to **satisfy** a linear homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}, \quad c_k \neq 0 \tag{8.32}$$

of order k with constant coefficients if $s_n = c_1 s_{n-1} + c_2 s_{n-2} + c_3 s_{n-3} + \cdots + c_k s_{n-k}$.

DEFINITION 8.2.4

If a sequence $s_0, s_1, s_2, \ldots, s_n, \ldots$ satisfies a linear homogeneous recurrence relation, then the sequence $s_0, s_1, s_2, \ldots, s_n, \ldots$ is also called a **solution** of that recurrence relation.

EXAMPLE 8.2.5

Consider the recurrence relation $a_n = 3a_{n-1}$. This is a linear homogeneous recurrence relation of order 1. Let t be a nonzero number and suppose $a_n = t^n$ for all $n \geq 0$. Then $a_n = 3a_{n-1}$ implies that $t^n = 3t^{n-1}$. Therefore, $t = 3$. Thus, we find that $a_n = 3^n$. Hence, the sequence $1, 3, 3^2, 3^3, \ldots 3^n, \ldots$ is a solution of the recurrence relation $a_n = 3a_{n-1}$.

Theorem 8.2.7: Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \quad c_2 \neq 0, \quad n > 1 \tag{8.34}$$

be a linear homogeneous recurrence relation with constant coefficients. Let t be a nonzero real number. Then the sequence $\{t^n\}$ satisfies the above recurrence relation if and only if

$$t^2 - c_1 t - c_2 = 0. \tag{8.35}$$

DEFINITION 8.2.8

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, $c_2 \neq 0$, $n > 1$ be a linear homogeneous recurrence relation with constant coefficients. The equation

$$t^2 - c_1 t - c_2 = 0$$

is called the **characteristic equation** of the recurrence relation.

Theorem 8.2.9: Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \quad n > 1 \tag{8.37}$$

be a linear homogeneous recurrence relation of order 2, where c_1 and c_2 are constants and $c_2 \neq 0$

(i) If the sequences $\{s_n\}$ and $\{p_n\}$ satisfy (8.37), then for any constants b and d, the sequence $\{bs_n + dp_n\}$ satisfies (8.37).

(ii) Let r be a root of the characteristic equation

$$t^2 - c_1 t - c_2 = 0 \tag{8.38}$$

of (8.37). Then the sequence $\{r^n\}$ is a solution of (8.37).

Theorem 8.2.10: Suppose that a sequence $\{a_n\}$ is a solution of the recurrence relation (8.37). If r_1 and r_2 are the distinct roots of the characteristic equation (8.38), then there exist constants b and d, which

Corollary 8.2.11: Suppose that

$$a_0 = d_0, \quad a_1 = d_1$$

are the initial conditions for the recurrence relation (8.37), where d_0 and d_1 are constants. Further suppose that r_1 and r_2 are the roots of (8.38). If $r_1 \neq r_2$, then there exist constants b and d, which are to be determined by initial conditions, such that the solution of the recurrence relation (8.37) is

$$a_n = br_1^n + dr_2^n, \quad n = 0, 1, \ldots.$$

Example 8.2.12

In this example, we solve the following linear homogeneous recurrence relation:

$$a_n = 7a_{n-1} - 10a_{n-2} \tag{8.41}$$

with initial conditions

$$a_0 = 1$$
$$a_1 = 8.$$

The characteristic equation of the given recurrence relation is:

$$t^2 - 7t + 10 = 0.$$

Next, we find the roots of this equation. Now,

$$t^2 - 7t + 10 = (t - 5)(t - 2)$$

and so

$$(t - 5)(t - 2) = 0.$$

This implies that the roots of the characteristic equation are $t = 5$, and $t = 2$. The roots are distinct. By Theorem 8.2.10, there exist constants c_1 and c_2, which are to be determined from initial conditions, such that

$$a_n = c_1 5^n + c_2 2^n, \quad n \geq 0.$$

We substitute $n = 0$ and $n = 1$, respectively, to obtain

$$a_0 = c_1 + c_2,$$
$$a_1 = 5c_1 + 2c_2.$$

Using the initial conditions, we get

$$c_1 + c_2 = 1,$$
$$5c_1 + 2c_2 = 8.$$

Solving these equations for c_1 and c_2, we get $c_1 = 2$ and $c_2 = -1$. Hence,

$$a_n = 2 \cdot 5^n - 2^n, \quad n \geq 0.$$

Hence, the sequence $\{2 \cdot 5^n - 2^n\}$ is the solution.

Theorem 8.2.13: Suppose that a sequence $\{s_n\}$ is a solution of the recurrence relation (8.37). If r_1 and r_2 are the roots of the characteristic equation (8.38) such that $r_1 = r_2 = r$, then there exist constants b and d, which are to be determined, such that the solution of the recurrence relation (8.37) is

$$s_n = br^n + dnr^n, \quad n = 0, 1, \ldots.$$

Corollary 8.2.14: Suppose that

$$a_0 = d_0, \qquad a_1 = d_1$$

are the initial conditions for the recurrence relation (8.37), where d_0 and d_1 are constants. Further suppose that r_1 and r_2 are the roots of (8.38) such that $r_1 = r_2 = r$. Then there exist constants b and d, which are to be determined from initial conditions, such that the solution of the recurrence relation (8.37) is

$$a_n = br^n + dnr^n, \quad n = 0, 1, \ldots.$$

EXAMPLE 8.2.15

In this example, we solve the following linear homogeneous recurrence relation:

$$a_n = 4a_{n-1} - 4a_{n-2}$$

with initial conditions

$$a_0 = 4$$
$$a_1 = 12.$$

The characteristic equation of this recurrence relation is the quadratic equation

$$t^2 - 4t + 4 = 0.$$

We find the roots of this equation. Now,

$$t^2 - 4t + 4 = (t - 2)(t - 2)$$

and so

$$(t - 2)(t - 2) = 0.$$

EXAMPLE 8.2.15

This implies that the roots of the characteristic equation are $t = 2$, and $t = 2$. The roots are not distinct. Therefore, by Theorem 8.2.13, there exist constants c_1 and c_2, which are to be determined from initial conditions, such that

$$a_n = c_1 2^n + c_2 n 2^n, \quad n = 0, 1, \ldots.$$

We substitute $n = 0$ and $n = 1$, respectively, to obtain

$$a_0 = c_1$$
$$a_1 = 2c_1 + 2c_2.$$

Using the initial conditions, we get

$$c_1 = 4,$$
$$2c_1 + 2c_2 = 12.$$

Solving these equations for c_1 and c_2, we get $c_1 = 4$ and $c_2 = 2$. Hence,

$$a_n = 4 \cdot 2^n + 2 \cdot n \cdot 2^n = 2 \cdot 2^{n+1} + n2^{n+1} = (2 + n)2^{n+1} = (n + 2)2^{n+1}, \quad n \geq 0.$$

Thus, we find that the sequence $\{(n + 2)2^{n+1}\}$ is the solution.

Theorem 8.2.16: Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}, \quad c_k \neq 0 \tag{8.42}$$

be a linear homogeneous recurrence relation with constant coefficients. Let t be a nonzero real number. Then the sequence $\{t^n\}$ is a solution of the above recurrence relation if and only if

$$t^n - c_1 t^{n-1} - c_2 t^{n-2} - c_3 t^{n-3} - \cdots - c_k t^{n-k} = 0.$$

DEFINITION 8.2.17

Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots + c_k a_{n-k}$, $c_k \neq 0$ be a linear homogeneous recurrence relation with constant coefficients. The equation

$$t^k - c_1 t^{k-1} - c_2 t^{k-2} - c_3 t^{k-3} - \cdots - c_k = 0$$

is called the **characteristic equation** of this linear homogeneous recurrence relation.

REMARK 8.2.18

To obtain the characteristic equation of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} + \cdots + c_ka_{n-k}$, $c_k \neq 0$, substitute $a_n = t^n, t \neq 0$, to get

$$t^n = c_1t^{n-1} + c_2t^{n-2} + c_3t^{n-3} + \cdots + c_kt^{n-k}.$$

Thus,

$$\begin{aligned} & t^n = c_1t^{n-1} + c_2t^{n-2} + c_3t^{n-3} + \cdots + c_kt^{n-k} \\ \Rightarrow\ & t^n - c_1t^{n-1} - c_2t^{n-2} - c_3t^{n-3} - \cdots - c_kt^{n-k} = 0 \\ \Rightarrow\ & t^{n-k}(t^k - c_1t^{k-1} - c_2t^{k-2} - c_3t^{k-3} - \cdots - c_k) = 0. \end{aligned}$$

Because $t \neq 0$, we have, $t^k - c_1t^{k-1} - c_2t^{k-2} - c_3t^{k-3} - \cdots - c_k = 0$, which is the characteristic equation.

Theorem 8.2.19: Let

$$a_n = c_1a_{n-1} + c_2a_{n-2} + c_3a_{n-3} + \cdots + c_ka_{n-k} \tag{8.44}$$

be a linear homogeneous recurrence relation of order k, where $c_k \neq 0$ and $c_1, c_2, c_3, \ldots,$ and c_k are constants. Let

$$t^k - c_1t^{k-1} - c_2t^{k-2} - c_3t^{k-3} - \cdots - c_k = 0$$

be the characteristic equation of (8.44).

(i) If the sequences $\{s_n\}_{n=0}^{\infty}$ and $\{p_n\}_{n=0}^{\infty}$ are solutions of (8.44), then for any constants b and d, the sequence $\{bs_n + dp_n\}_{n=0}^{\infty}$ is a solution of (8.44).

(ii) If r is a root of the characteristic equation, then the sequence $1, r, r^2, \ldots, r^n, \ldots$ is a solution of (8.44).

(iii) If $r_1, r_2, \ldots, r_k$ are distinct roots of the characteristic equations, then there exist constants $b_1, b_2, b_3, \ldots, b_k$, which are to be determined from initial conditions, such that a solution of (8.44) is given by

$$a_n = b_1r_1^n + b_2r_2^n + b_3r_3^n + \cdots + b_kr_k^n,$$

(iv) If r is a root, of multiplicity m, of the characteristic equation, then $a_n = r^n$, $a_n = nr^n$, $a_n = n^2r^n, \ldots$, and $a_n = n^{m-1}r^n$ are solutions of (8.44).

(v) Suppose that

$$a_0 = d_0, a_1 = d_1, \ldots, a_{n-1} = d_{n-1}$$

are the initial conditions for the recurrence relation (8.44), where $d_0, d_1, \ldots$, and d_{n-1} are constants. If $r_1, r_2, \ldots$, and r_t are t distinct roots of the characteristic equation with multiplicities $m_1, m_2, \ldots, m_t$ and $m_1 + m_2 + \cdots + m_t = k$, then there exist constants c_{ij}, which are to be determined from the initial conditions, such that the solution of the recurrence relation (8.44) is

$$\begin{aligned} a_n = (c_{00} + c_{01}n + \cdots + c_{0m_1}n^{m_1-1})r_1^n \\ + (c_{10} + c_{11}n + \cdots + c_{1m_2}n^{m_2-1})r_2^n \\ + \cdots + (c_{t0} + c_{t1}n + \cdots + c_{tm_t}n^{m_t-1})r_t^n, \quad n = 0, 1, \ldots . \end{aligned}$$

A **linear nonhomogeneous recurrence relation** with constants coefficients is a recurrence relation of the form

$$a_n + c_1a_{n-1} + \cdots + c_ka_{n-k} = f(n), \tag{8.55}$$

where $c_i, i = 1, 2, \ldots, k$, are constants, $c_k \neq 0$, and $f(n)$ is a nonzero real-valued function.

If $f(n) = 0$, then (8.55) is a linear homogeneous equation (which we discussed in the previous section). There is no known general method for solving nonhomogeneous linear recurrence equations. However, we can develop a method for solving the special case

$$c_0a_n + c_1a_{n-1} + \cdots + c_ka_{n-k} = b^np(n), \tag{8.56}$$

where b is a constant and $p(n)$ is a polynomial in n.

Linear Nonhomogenous Recurrence Relations

EXAMPLE 8.3.2

Consider the recurrence

$$a_n + 5a_{n-1} + 6a_{n-2} = 3^n.$$

This is a nonhomogeneous recurrence relation of the form (8.56). Here $k = 2$, $b = 3$, and $p(n) = 1$.

EXAMPLE 8.3.3

Consider the recurrence

$$a_n + 5a_{n-1} + 6a_{n-2} = 3^n(n^2 + 6n + 5).$$

This is a nonhomogeneous recurrence relation of the form (8.56). Here $k = 2$, $b = 3$, and $p(n) = n^2 + 6n + 5$.

Linear Nonhomogenous Recurrence Relations

Theorem 8.3.5: Let

$$a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = f(n) \tag{8.62}$$

be a nonhomogeneous recurrence relation, where c_i, $i = 1, 2, \ldots, k$, are constants, $c_k \neq 0$, and $f(n)$ is a nonzero real-valued function. Suppose $\{r_n\}$ is a particular solution of (8.62). Then $\{u_n\}$ is a solution of (8.62) if and only if $u_n = r_n + s_n$, for all n, and $\{s_n\}$ is a solution of the associated homogeneous part, $a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = 0$.

Theorem 8.3.6: Let

$$a_n - da_{n-1} = b^n u, \quad n \geq 1 \tag{8.67}$$

be a nonhomogeneous linear recurrence relation, with the initial condition

$$a_0 = e_0, \tag{8.68}$$

where d, b, u, and e_0 are constants, and b and u are nonzero. This nonhomogeneous linear recurrence relation can be transformed into the following linear homogeneous recurrence relation:

$$a_n - (b + d)a_{n-1} + bda_{n-2} = 0, \quad n \geq 2$$

with the initial conditions $a_0 = e_0$ and $a_1 = de_0 + bu$.

Moreover,

(i) if $b \neq d$, then there exists a constant c_0, which is to be determined from the initial condition, such that

$$a_n = c_0 d^n + \left(\frac{bu}{b - d}\right) b^n.$$

(ii) if $b = d$, then there exists a constant c_0, which to be is determined from the initial condition, such that

$$a_n = c_0 b^n + unb^n.$$

EXAMPLE 8.3.7

In this example, we use Theorem 8.3.6 to solve the recurrence relation

$$a_n - 4a_{n-1} = 8^n, \quad n \geq 1,$$

with the initial condition

$$a_0 = 1.$$

This is a recurrence relation of the form

$$a_n - da_{n-1} = b^n u,$$

where $d = 4$, $b = 8$, and $u = 1$. Because $b \neq d$,

$$\begin{aligned} a_n &= c_0 d^n + \frac{bu}{b - d} b^n \\ &= c_0 4^n + \frac{8}{4} 8^n \\ &= c_0 4^n + 2 \cdot 8^n \end{aligned}$$

for all $n \geq 0$, where c_0 is a constant satisfying the initial condition.

Now

$$1 = a_0 = c_0 4^0 + 2 \cdot 8^0 = c_0 + 2.$$

Hence, $c_0 = -1$. This implies that $a_n = -1 \cdot 4^n + 2 \cdot 8^n$ for all $n \geq 0$.

Theorem 8.3.10: Let

$$a_n - da_{n-1} = b^n(un + v), \quad n \geq 1, \tag{8.83}$$

be a nonhomogeneous linear recurrence relation, with the initial condition

$$a_0 = e_0, \tag{8.84}$$

where d, b, u, v, and e_0 are constants, and b and u are nonzero. This nonhomogeneous linear recurrence relation can be transformed into the following linear homogeneous recurrence relation:

$$a_n - (2b + d)a_{n-1} + b(2d + b)a_{n-2} - b^2 da_{n-3} = 0, \quad n \geq 3 \tag{8.85}$$

with the initial conditions

$$a_0 = e_0 \quad \text{and} \quad a_1 = de_0 + b(u + v).$$

Moreover, the characteristic equation of (8.85) is

$$(t - d)(t - b)^2 = 0. \tag{8.86}$$

Let $\{r_n\}$ be a solution of (8.83).

(i) Suppose $b \neq d$. Then r_n is of the form

$$r_n = c_0 d^n + c_1 b^n + c_2 nb^n,$$

where c_0, c_1, and c_2 are some constants.

(ii) Suppose $b = d$. Then $\{r_n\}$ is of the form

$$r_n = c_0 b^n + c_1 nb^n + c_2 n^2 b^n,$$

where c_0, c_1, and c_2 are some constants.

EXAMPLE 8.3.11

Consider the recurrence relation

$$a_n - 3a_{n-1} = 2^n(4n + 3), \quad n > 1 \tag{8.94}$$

with initial conditions

$$a_0 = 0,$$
$$a_1 = 14.$$

This is a recurrence relation of the form

$$a_n - da_{n-1} = b^n(un + v).$$

Here $d = 3$, $b = 2$, $u = 4$, and $v = 3$.

We can solve this recurrence by using the technique of Theorem 8.3.10 and obtaining

$$a_n = c_0 3^n + c_1 2^n + c_2 n 2^n,$$

where c_0, c_1, and c_2 are constants, which are to be determined from the initial conditions.

EXAMPLE 8.3.11

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We can solve this recurrence by using the technique of Theorem 8.3.10 and obtaining

$$a_n = c_0 3^n + c_1 2^n + c_2 n 2^n,$$

where c_0, c_1, and c_2 are constants, which are to be determined from the initial conditions.

Put $n = 2$ in (8.92) to get

$$a_2 - 3a_1 = 2^2(4 \cdot 2 + 3) = 44.$$

Because $a_1 = 14$, we get

$$a_2 = 3 \cdot 14 + 44 = 86.$$

Thus,

$$\begin{aligned} a_0 &= c_0 + c_1 = 0 \\ a_1 &= c_0 \cdot 3 + c_1 \cdot 2 + c_2 \cdot 2 = 14 \\ a_2 &= c_0 \cdot 3^2 + c_1 \cdot 2^2 + c_2 \cdot 2 \cdot 2^2 = 86 \end{aligned}$$

This implies that

$$\begin{aligned} c_0 + c_1 &= 0 \\ 3c_0 + 2c_1 + 2c_2 &= 14 \\ 9c_0 + 4c_1 + 8c_2 &= 86 \end{aligned}$$

We solve these equations for c_0, c_1, and c_2 to obtain $c_0 = 30$, $c_1 = -30$, and $c_2 = -8$. Thus, we find that

$$a_n = 30(3^n) - 30(2^n) - n2^{n+3}, \quad n \geq 0. \tag{8.95}$$

Theorem 8.3.13: Let

$$a_n + d_1 a_{n-1} + \cdots + d_k a_{n-k} = b^n p(n) \tag{8.96}$$

be a nonhomogeneous linear recurrence relation, where $p(n)$ is a polynomial of degree m. Then from this nonhomogeneous linear recurrence relation we can obtain a linear homogeneous recurrence that has following characteristic equation:

$$(t^k + d_1 t^{k-1} + \cdots + d_k)(t - b)^{m+1} = 0. \tag{8.97}$$

Moreover, a solution of (8.96) is also a solution of the linear homogeneous recurrence whose characteristic equation is given by (8.97).

Linear Recurrences

There is a class of recurrence relations which *can* be solved analytically in general. These are called *linear recurrences* and include the Fibonacci recurrence.
Begin by showing how to solve Fibonacci:

Solving Fibonacci

Recipe solution has 3 basic steps:

1) Assume solution of the form $an = r\ n$
2) Find all possible r's that seem to make this work. Call these1 $r1$ and $r2$. Modify assumed solution to ***general solution*** $an = Ar1n + Br2n$ where A,B are constants.
3) Use initial conditions to find A,B and obtain specific solution.

Solving Fibonacci

1) Assume exponential solution of the form $an = r\ n$:

   Plug this into $a_n = a_{n-1} + a_{n-2}$:

   $$r^n = r^{n-1} + r^{n-2}$$

   Notice that all three terms have a common r^{n-2} factor, so divide this out:

   $r^n / r^{n-2} = (r^{n-1} + r^{n-2})/r^{n-2} \Rightarrow r^2 = r + 1$

This equation is called the ***characteristic equation*** of the recurrence relation.

2) Find all possible r's that solve characteristic

   $$r^2 = r + 1$$

   Call these r_1 and r_2.[1] General solution is

   $$a_n = Ar_1^n + Br_2^n \text{ where } A,B \text{ are constants.}$$

Quadratic formula2 gives:

$$r = (1 \pm \sqrt{5})/2$$

So $r_1 = (1+\sqrt{5})/2$, $r_2 = (1-\sqrt{5})/2$

General solution:

$$a_n = A\,[(1+\sqrt{5})/2]^n + B\,[(1-\sqrt{5})/2]^n$$

Solving Fibonacci

◆ Use initial conditions $a_0 = 0$, $a_1 = 1$ to find A,B and obtain specific solution.

$0=a_0 = A\,[(1+\sqrt{5})/2]^0 + B\,[(1-\sqrt{5})/2]^0 = A + B$

$1=a_1 = A\,[(1+\sqrt{5})/2]^1 + B\,[(1-\sqrt{5})/2]^1 = A(1+\sqrt{5})/2 + B\,(1-\sqrt{5})/2$

$= (A+B)/2 + (A-B)\sqrt{5}/2$

First equation give $B = -A$. Plug into 2[nd]:

$1 = 0 + 2A\sqrt{5}/2$ so $A = 1/\sqrt{5}$, $B = -1/\sqrt{5}$

Final answer:

(CHECK IT!)

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Linear Recurrences with Constant Coefficients

Previous method generalizes to solving "*linear recurrence relations with constant coefficients*":

DEF: A recurrence relation is said to be ***linear*** if *an* is a linear combination of the previous terms plus a function of *n*. I.e. no squares, cubes or other complicated function of the previous *ai* can occur. If in addition all the coefficients are constants then the recurrence relation is said to have ***constant coefficients***.

Linear Recurrences with Constant Coefficients

Q: Which of the following are linear with constant coefficients?

1. $a_n = 2a_{n-1}$
2. $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$
3. $a_n = a_{n-1}^{\;2}$
4. Partition function: $$p_n = \sum_{i=0}^{n-1} p_i \cdot C(n-1, n-1-i)$$

Linear Recurrences with Constant Coefficients

A:

1. $a_n = 2a_{n-1}$: YES
2. $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$: YES
3. $a_n = a_{n-1}^2$: NO. Squaring is not a linear operation. Similarly $a_n = a_{n-1}a_{n-2}$ and $a_n = \cos(a_{n-2})$ are non-linear.
4. Partition function: $p_n = \sum_{i=0}^{n-1} p_i \cdot C(n-1, n-1-i)$ NO.

   This is linear, but coefficients are not constant as $C(n-1, n-1-i)$ is a non-constant function of n.

Homogeneous Linear Recurrences

To solve such recurrences we must first know how to solve an easier type of recurrence relation:

DEF: A linear recurrence relation is said to be ***homogeneous*** if it is a linear combination of the previous terms of the recurrence *without* an additional function of n.

Q: Which of the following are homogeneous?

1. $a_n = 2a_{n-1}$
2. $a_n = 2a_{n-1} + 2_{n-3} - a_{n-3}$
3. Partition function: $p_n = \sum_{i=0}^{n-1} p_i \cdot C(n-1, n-1-i)$

Linear Recurrences with Constant Coefficients

A:

1. $a_n = 2a_{n-1}$: YES
2. $a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$: No. There's an extra term $f(n) = 2^{n-3}$
3. Partition function:

$$p_n = \sum_{i=0}^{n-1} p_i \cdot C(n-1, n-1-i)$$

YES. No terms appear not involving the previous p_i

Homogeneous Linear Recurrences with Const. Coeff.'s

The 3-step process used for the Fibonacci recurrence works well for general homogeneous linear recurrence relations with constant coefficients. There are a few instances where some modification is necessary.

Homogeneous – Complications

1) *Repeating roots* in characteristic equation. Repeating roots imply that don't learn anything new from second root, so may not have enough information to solve formula with given initial conditions. We'll see how to deal with this on next slide.
2) *Non-real number roots* in characteristic equation. If the sequence has periodic behavior, may get complex roots (for example $a_n = -a_{n-2}$)[1]. We won't worry about this case (in principle, same method works as before, except use complex arithmetic).

Complication: Repeating Roots

EG: Solve $a_n = 2a_{n-1}-a_{n-2}$, $a_0 = 1$, $a_1 = 2$
Find characteristic equation by plugging in $a_n = r^n$:

$$r^2 - 2r + 1 = 0$$

Since $r^2 - 2r + 1 = (r-1)^2$ the root $r = 1$ repeats.
If we tried to solve by using general solution

$$a_n = Ar_1^n + Br_2^n = A1^n + B1^n = A+B$$

which forces a_n to be a constant function (→←).
SOLUTION: Multiply second solution by n so general solution looks like:

$$a_n = Ar_1^n + Bnr_1^n$$

Complication: Repeating Roots

Solve $a_n = 2a_{n-1}-a_{n-2}$, $a_0 = 1$, $a_1 = 2$
General solution: $a_n = A1^n + Bn1^n = A+Bn$

Plug into initial conditions
$1 = a_0 = A+B\cdot 0\cdot 1^0 = A$
$2 = a_0 = A\cdot 1^1 + B\cdot 1\cdot 1^1 = A+B$
Plugging first equation $A = 1$ into second: $2 = 1+B$ implies $B = 1$.
Final answer: $a_n = 1+n$

(CHECK IT!)

The Nonhomogeneous Case

Consider the Tower of Hanoi recurrence (see Rosen p. 311-313)
$a_n = 2a_{n-1}+1$.
Could solve using telescoping. Instead let's solve it methodically. Rewrite:
$a_n - 2a_{n-1} = 1$

1) Solve with the RHS set to 0, i.e. solve the homogeneous case.
2) Add a particular solution to get general solution. I.e. use rule:

General Nonhomogeneous	=	General homogeneous	+	Particular Nonhomogeneous

The Nonhomogeneous Case

$$a_n - 2a_{n-1} = 1$$

1) Solve with the RHS set to 0, i.e. solve

$$a_n - 2a_{n-1} = 0$$

Characteristic equation: $r - 2 = 0$
so unique root is $r = 2$. General solution to homogeneous equation is

$$a_n = A \cdot 2^n$$

The Nonhomogeneous Case

2) Add a particular solution to get general solution for $a_n - 2a_{n-1} = 1$.
Use rule:

General Nonhomogeneous	=	General homogeneous	+	Particular Nonhomogeneous

There are little tricks for guessing particular nonhomogeneous solutions.
For example, when the RHS is constant, the guess should also be a constant.[1]
So guess a particular solution of the form $b_n=C$.
Plug into the original recursion:
$1 = b_n - 2b_{n-1} = C - 2C = -C$. Therefore $C = -1$.
General solution: $a_n = A \cdot 2^n - 1$.

The Nonhomogeneous Case

Finally, use initial conditions to get closed solution. In the case of the Towers of Hanoi recursion, initial condition is:
$a_1 = 1$
Using general solution $an = A\cdot 2^n - 1$ we get:
$1 = a_1 = A\cdot 2^1 - 1 = 2A - 1$.
Therefore, $2 = 2A$, so $A = 1$.
Final answer: $a_n = 2^n - 1$

More Complicated

EG: Find the general solution to recurrence from the bit strings example:

$$a_n = 2a_{n-1} + 2^{n-3} - a_{n-3}$$

1) Rewrite as $a_n - 2a_{n-1} + a_{n-3} = 2^{n-3}$ and solve homogeneous part:

Characteristic equation: $r^3 - 2r + 1 = 0$.

Guess root $r = \pm 1$ as integer roots divide.
$r = 1$ works, so divide out by $(r - 1)$ to get
$r^3 - 2r + 1 = (r - 1)(r^2 + r - 1)$.

More Complicated

$r^3 - 2r + 1 = (r - 1)(r^2 + r - 1)$.
Quadratic formula on $r^2 + r - 1$:

$$r = (-1 \pm \sqrt{5})/2$$

So $r_1 = 1$, $r_2 = (-1+\sqrt{5})/2$, $r_3 = (-1-\sqrt{5})/2$
General homogeneous solution:

$$a_n = A + B\,[(-1+\sqrt{5})/2]^n + C\,[(-1-\sqrt{5})/2]^n$$

More Complicated

2) Nonhomogeneous particular solution to $a_n - 2a_{n-1} + a_{n-3} = 2^{n-3}$
Guess the form $b_n = k\,2^n$. Plug guess in:

$$k\,2^n - 2k\,2^{n-1} + k\,2^{n-3} = 2^{n-3}$$

Simplifies to: $k = 1$.
So particular solution is $b_n = 2^n$

General Nonhomogeneous = General homogeneous + Particular Nonhomogeneous

Final answer:
$a_n = A + B\,[(-1+\sqrt{5})/2]^n + C\,[(-1-\sqrt{5})/2]^n + 2^n$